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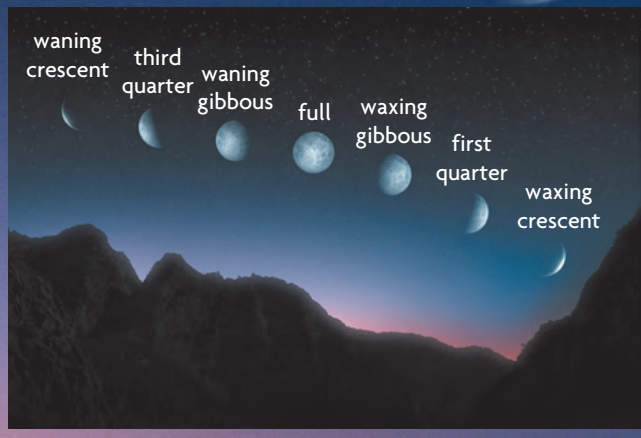
Chapter

8

Sinusoidal Functions



LEARNING GOALS

You will be able to develop your algebraic and graphical reasoning skills by

- Describing the characteristics of sinusoidal functions using their graphs and equations
- Graphing data for a situation and modelling the situation using a sinusoidal function
- Solving problems using sinusoidal function models

? The Moon appears in a cycle of eight phases. A different percent of the Moon is illuminated during each phase, except the new phase. No part of the Moon is illuminated during the new phase. This cycle repeats about every 29.5 days.

Suppose that you were asked to draw a graph to represent the percent of the Moon that is illuminated during each phase, for a full year. What would this graph look like?

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8 Getting Started

YOU WILL NEED

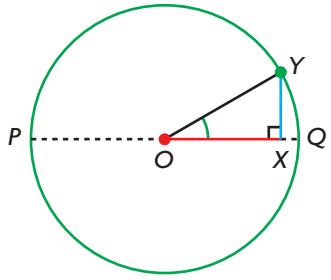
- dynamic geometry software OR compass, ruler, and protractor

Sine and Cosine Patterns

Recall that you have worked with **trigonometric ratios** involving angles and sides in right, acute, and obtuse triangles.

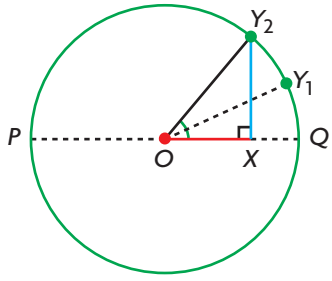
? How are the sine and cosine ratios similar, and how they are different?

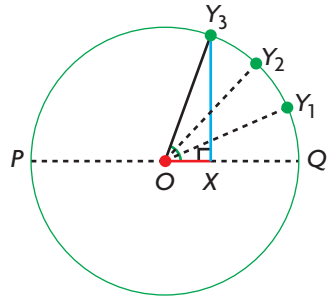
A. Draw a circle with a radius of 5 cm centred at O. Draw diameter PQ and radius OY. Construct line segment YX, perpendicular to PQ at X. Your diagram should look similar to the one shown.



B. Measure OX, YX, $\angle YOX$, and $\angle YOQ$. Determine the sine and cosine of both angles.

C. Gradually increase the measure of $\angle YOQ$ toward 90° by moving point Y along the circumference of the circle. Repeat step B for several different angle measures.





As the measure of $\angle YOQ$ increases, what happens to the following values?

- the length of YX
- the length of OX
- the value of sin $\angle YOQ$
- the value of cos $\angle YOQ$

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D. As the measure of $\angle YOX$ gets very close to 90°, what happens to these values?
- the length of YX
- the value of $\sin \angle YOX$
- the length of OX
- the value of $\cos \angle YOX$

E. Predict the values of sin 90° and cos 90°. Provide your reasoning.

F. Describe the pattern in the values of
- $\sin x$ as the measure of angle x increases from 0° to 90° and
- $\cos x$ as the measure of angle x increases from 0° to 90°.

G. How are the sine and cosine ratios the same? How are they different?

WHAT DO You Think?

Decide whether you agree or disagree with each statement. Explain your decision.

1. There are many different angles that have the same sine ratio.
2. The sine of an angle can be negative.
3. The sine ratio can never be greater than 1.
4. Functions that involve the sine and cosine ratios can be used to predict naturally occurring events.

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8.1 Understanding Angles

YOU WILL NEED

- compass
- pipe cleaners
- protractor

GOAL

Estimate and determine benchmarks for angle measure.

INVESTIGATE the Math

Abner is a carpenter. When he needs to cut a hole in wood, he attaches a hole saw to a drill. For one project, he used hole saws with diameters of 10 cm, 14 cm, and 20 cm. Abner knows that when the diameter of the saw increases, the circumference of the hole also increases.

? How many times will the radius of each hole saw fit around the circumference of the circle it cuts?

A. Work in groups of three, with each member drawing a circle to represent one of the three different hole saws. Label the radius of your circle *CA*.

B. Cut a pipe cleaner to a length equal to radius *CA*.

C. Place and bend the pipe cleaner around the circumference of your circle, beginning at point *A* and ending at point *P*, as shown by the red arc.

D. Draw radius *CP*.

E. Measure $\angle PCA$ with a protractor. What is the measure of $\angle PCA$ in degrees? Is the measure of $\angle PCA$ the same for all three circles?

F. Use the pipe cleaner to continue marking radius lengths all the way around your circle. About how many radius lengths are there in one complete circumference of your hole saw? Compare your results with those of your group members.

Reflecting

G. The circumference, C, of a circle is given by the formula

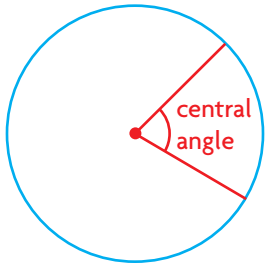
$$C = 2\pi r$$

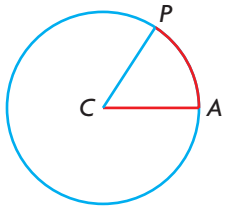
where r represents the radius. Explain how this formula relates to your answer for part F.

H. What degree measure is equal to 2π **radians**?

EXPLORE...

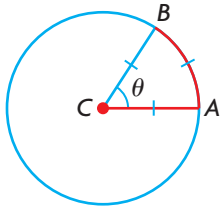
- Devise a method to estimate the measure of any central angle in a circle without using a protractor. Explain how your method works.





radian

The measure of the central angle of a circle subtended by an arc that is the same length as the radius of the circle.



$\angle\theta = 1$

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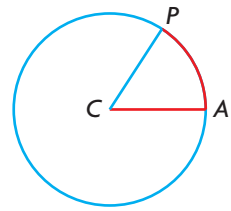
APPLY the Math

EXAMPLE 1 Expressing 1 radian in degrees

The measure of $\angle PCA$ is 1 radian. Calculate the measure of $\angle PCA$ in degrees.



Communication *Tip*

Angles can be measured using different units. These include degrees, radians, gradients, and minutes and seconds.

Thao's Solution

One complete revolution of the circle measures 360°.
One complete revolution of the circle also measures 2π in radians.

2π radians $= 360°$

So π radians must be equal to $\frac{360°}{2}$.

π radians $= 180°$

To determine the measure of 1 radian, I can divide both sides of the equation by π:

$$1 \text{ radian} = \frac{180°}{\pi}$$

$$1 \text{ radian} = 57.295…°$$

$\angle PCA$ measures about 57.3°.

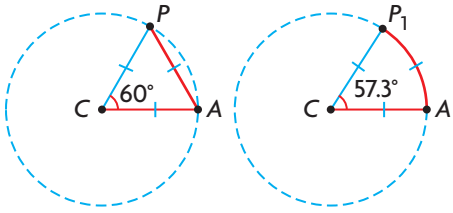
I knew the total number of degrees in a circle and the total number of radians in a circle.

I determined the relationship between degrees and radians using the transitive property.

The value of π is 3.141…, and 180° is equivalent to π, so I know that 180° is equivalent to just over 3 radians. So it makes sense that 1 radian is just under 60°.

To visualize 1 radian, use an equilateral triangle with a flexible side, AP, drawn inside a circle.

I imagined moving the point P along the circumference of the circle, forcing the chord AP to bend, until it fit along the circumference, from point A to point P_1.



Since side PC is moved toward side AC, the angle in the second diagram must measure a bit less than 60°.

Therefore, 60° can be used as a benchmark for 1 radian.

Your Turn

Suggest two more benchmarks you could use to estimate the size of an angle given in radian measure.

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EXAMPLE 2 Estimating values of angles in radian measure

Estimate the value of each angle in radian measure.

a) 90° **b)** 45° **c)** 150°

Natalia's Solution

a) 90° is half of 180°.
Since 180° is equivalent to π in radian measure, I can use this relationship to estimate 90° in radians. I can round π to 3.2 so I can do mental calculations more easily.

I can estimate 90° as $\frac{3.2}{2}$, or 1.6.

I decided to relate 90° to the benchmark angle of 180°.

Communication *Tip*

Any angle measures presented as real numbers without units are considered to be in radians.

b) 45° is half of 90°, so 45° must be about half of 1.6.
I can estimate 45° as $\frac{1.6}{2}$, or 0.8.

I can use the benchmark that 1 radian is slightly less than 60° to estimate the measure of 30° in radians.

I divided by 2 again to determine my estimate.

I knew that my estimate was high, because I rounded π up to 3.2.

c) 150° is 30° less than 180°.
I can estimate 30° as about 0.5 radians.

I can round π to 3.2 so I can easily do mental calculations.

180° is equivalent to π in radian measure.
I can estimate 150° as $3.2 - 0.5$, or 2.7.

I knew that my estimate was a bit high, because I rounded π up to 3.2.

Your Turn

Estimate the value of each angle in radian measure.

a) 120° **b)** 135°

Communication *Tip*

It is possible to get different estimates of angle measures. Your estimates may vary depending on which benchmarks you use.

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EXAMPLE 3 Estimating angles greater than 180° in radian measure

Estimate the value of each angle in radian measure.

a) 240° **b)** 450° **c)** 690°

Sheila's Solution

a) I thought of 240° as the sum of 180° and 60°.
180° is slightly less than 3.2 radians.
60° is slightly more than 1 radian.
I estimate 240° is equivalent to about 4.2 radians.

I knew that 1 radian is slightly less than 60°.

b) I knew that 450° is 90° more than 360°.
360° is about 6.3 radians.
90° is about 1.6 radians.
I estimate 450° is slightly less than 7.9 radians.

I knew that my estimate was slightly high, because I rounded up the radian equivalents for both 360° and 90°.

c) Two complete revolutions of the circle measure 720°.
690° is 30° less than 720°.
30° is half of 60°, so it is about 0.5 in radian measure.
Two complete revolutions are about $2 \cdot 6.3$, or 12.6 radians.
690° is about $12.6 - 0.5$, or 12.1 radians.

I knew that 60° is about 1 in radian measure.

I knew that 1 complete revolution measures 2π radians, which is about 6.3.

I estimated 30° in radian measure. Then I subtracted this value from two revolutions.

Your Turn

Estimate the value of each angle in radian measure.

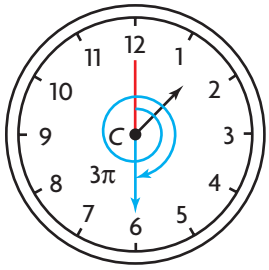
a) 420° **b)** 495° **c)** 660°

EXAMPLE 4 Comparing angles in radian measure

Determine which angle is larger: 3π or 8.

Xavier's Solution: Using benchmarks and visualization

3π is equivalent to $1\frac{1}{2}$ revolutions.



I knew that π represents $\frac{1}{2}$ revolution and 2π represents 1 revolution.

I visualized $1\frac{1}{2}$ revolutions using the minute hand of a clock. Starting at noon, $1\frac{1}{2}$ revolutions is equivalent to 1:30.

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1 revolution is about 6.3 in radian measure.

$\frac{1}{4}$ revolution is about 1.6 radians.

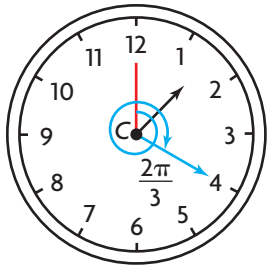
Therefore, $1\frac{1}{4}$ revolutions is about 7.9 radians.

I used benchmarks to estimate the position of the second angle, 8. I knew that $\frac{1}{2}$ revolution, 180°, is π in radian measure, or about 3.2. I divided by 2 to get an estimate for $\frac{1}{4}$ revolution, 90°.

I knew that my estimate for $1\frac{1}{4}$ revolutions was slightly high, because I rounded up my radian estimates for both 1 revolution and $\frac{1}{4}$ revolution.

8 is a little greater than 7.9.

So, 8 is a little greater than $1\frac{1}{4}$ revolutions.



I visualized an angle that is a little greater than $1\frac{1}{4}$ revolutions on a clock, which is equivalent to 1:15 if I started at noon. The minute hand should be close to about 4.

$1\frac{1}{2}$ revolutions is greater.

Therefore, 3π is larger than 8.

My answer makes sense since 3π is about 3(3.14), or 9.42, in radian measure.

Zachary's Solution: Expressing the angles in degrees

$\pi = 180°$

$2\pi = 360°$

I knew the degree equivalents for π and 2π radians.

$3\pi = 180° + 360°$

$3\pi = 540°$

8 is equivalent to about $8 \cdot 60°$, or 480°.

I knew that 1 in radian measure is about 60°, so 8 in radian measure is about 8 times 60°.

I knew that my estimate was high, because 1 is slightly less than 60°.

$540° > 480°$

3π is larger than 8.

Your Turn

Which solution do you prefer: Xavier's or Zachary's? Explain.

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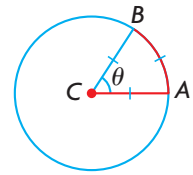
In Summary

Key Ideas

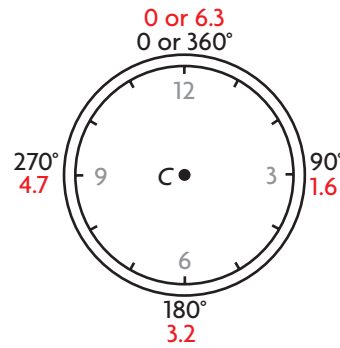
- Radian measure is an alternative way to express the size of an angle.
- Using radians allows you to express the measure of an angle as a real number without units.
- The central angle formed by one complete revolution in a circle is 360°, or 2π in radian measure.

Need to Know

- Use benchmarks to estimate the degree measure of an angle given in radians.
- In radian measure,
  - 1 is equivalent to about 60°;
  - π is equivalent to 180°;
  - 2π is equivalent to 360°.
- Decimal approximations can be used for benchmarks to visualize the approximate size of an angle measured in radians.



$\angle\theta = 57.3°$ or 1 in radian measure



Some estimation benchmarks

CHECK Your Understanding

1. Sketch an angle with each degree measure, and then estimate the measure in radians. Estimate to the nearest tenth.

a) 45° **b)** 150° **c)** 180°

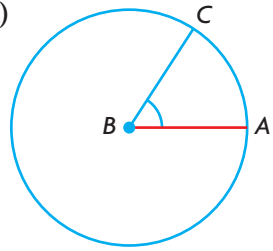
2. Estimate the value of each radian measure in degrees. Estimate to the nearest degree.

a) 1.6 **b)** 0.5 **c)** 2.4 **d)** 4.7

PRACTISING

3. Estimate, to the nearest degree, the measure of each central angle. Check your estimate with a protractor.

a)



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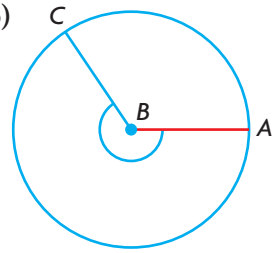

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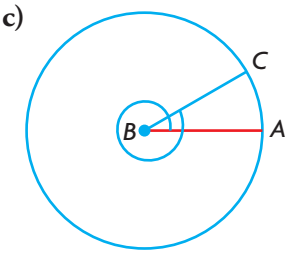

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4. Sketch an angle with each given measure, and then estimate, to the nearest tenth, the equivalent measure in radians.
a) 400° **b)** 640°

5. Estimate, to the nearest degree, the equivalent measure in degrees.
a) 8.1 **b)** 10.5

6. Imagine that it is now 9 a.m.
a) What time will it be when the minute hand has rotated through each of the following angles?
i) 120° **ii)** 330° **iii)** 690°
b) Estimate, to the nearest tenth, each angle measure in radian measure.

7. Imagine that you are standing on the circumference of a circle that has a radius of 3 m. You move a third of the way around the circle.
a) How far did you travel?
b) What central angle was created? Express the measure in degrees and radians.

8. For each pair of angle measures, determine which measure is greater.
a) 100°, 2 **b)** 45°, 0.5 **c)** 280°, 5 **d)** 400°, 6.5

9. Nim claims that any central angle, measured in radians, in a circle with a radius of 5 m will be half the measure of an equivalent angle in a circle with a radius of 10 m. Do you agree or disagree? Explain.

Closing

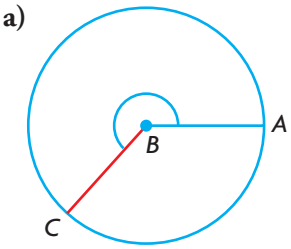
10. Explain how you could estimate the radian measure equivalent of an angle greater than 360°. Give three examples. Provide two different estimation strategies for each example.

Extending

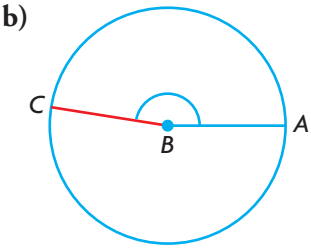
11. Sketch an angle with each measure.
a) 1200° **b)** 29.3

12. Estimate, to the nearest tenth, the measure of each central angle in radians. Check your estimate by measuring the angle with a protractor.

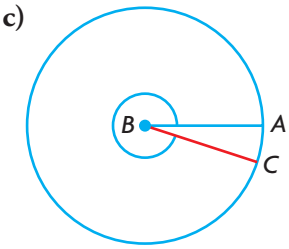
a)



b)



c)



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8.2 Exploring Graphs of Periodic Functions

GOAL

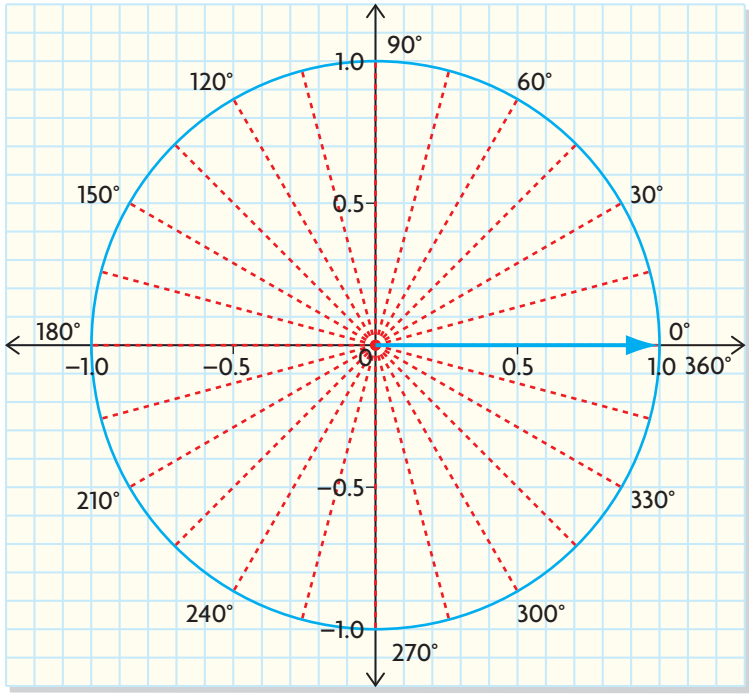
Investigate the characteristics of the graphs of sine and cosine functions.

YOU WILL NEED

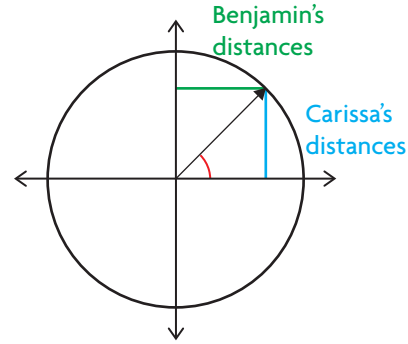
- ruler
- templates
- graphing technology

EXPLORE the Math

Carissa and Benjamin created a spinner. They glued graph paper to a board and centred the pointer at the origin. They turned the pointer counterclockwise through two complete turns.



Carissa graphed the distance from the tip of the pointer to the horizontal axis as a function of the angle, x, it had rotated. Benjamin graphed the distance from the tip of the pointer to the vertical axis as a function of the angle, x, it had rotated.



The students compared their graphs and examined:

- the number of x-intercepts
- the y-intercept
- the domain
- the range

? What would Carissa's and Benjamin's graphs look like for two complete spins, and what are the characteristics of these graphs?

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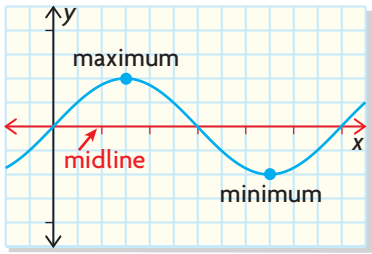
periodic function

A function whose graph repeats in regular intervals or cycles.

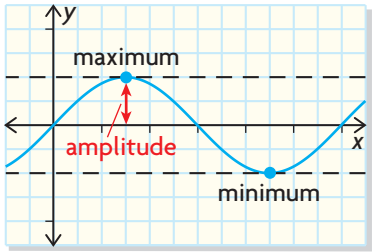
midline

The horizontal line halfway between the maximum and minimum values of a periodic function.



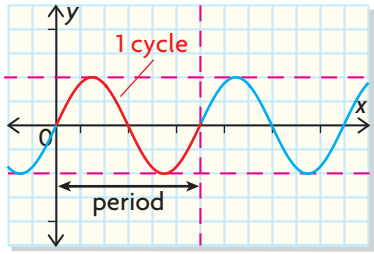
amplitude

The distance from the midline to either the maximum or minimum value of a periodic function; the amplitude is always expressed as a positive number.



period

The length of the interval of the domain to complete one cycle.



Reflecting

A. Based on your observations, which characteristics of the graphs of periodic functions are similar to characteristics of the graphs of polynomial, exponential, and logarithmic functions you have studied?

B. Based on your observations, which characteristics of the graphs of periodic functions differ from characteristics of the graphs of polynomial, exponential, and logarithmic functions you have studied?

C. Benjamin claims that if he continued drawing his graph for several more revolutions, he would notice a repeating pattern. This repeating pattern would result in a **periodic function** with several more x-intercepts. Do you agree or disagree? Explain.

D. Carissa claims that if she continued drawing her graph for several more complete revolutions, the range would remain the same. Do you agree or disagree? Explain.

E. Describe the relationships among the range, the **midline**, and the **amplitude** of each graph.

F. What is the **period** of each graph? Explain how the period helps to describe a characteristic of the graph of a periodic function.

G. Ensure that your graphing technology is in degree mode, then graph the periodic function

$$y = \sin x$$

for the domain $\{x \mid 0° \leq x \leq 720°, x \in \mathrm{R}\}$. How does this graph compare with Carissa's graph? Explain the reasons for any similarities or differences.

H. Graph the periodic function

$$y = \cos x$$

for the domain $\{x \mid 0° \leq x \leq 720°, x \in \mathrm{R}\}$ you used in part G. How does this graph compare with Benjamin's graph? Explain the reasons for any similarities or differences.

I. Ensure that your graphing technology is in radian mode, then graph

$$y = \sin x \text{ and}$$
$$y = \cos x$$

for the domain $\{x \mid 0 \leq x \leq 4\pi, x \in \mathrm{R}\}$. Compare these graphs with the graphs you drew in parts G and H.

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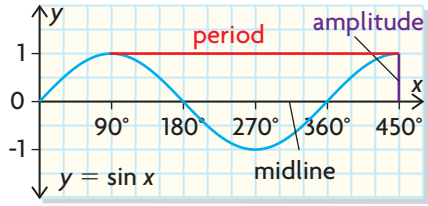
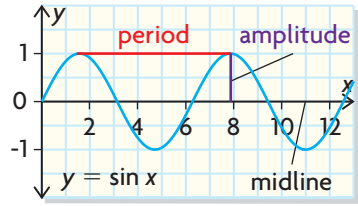
In Summary

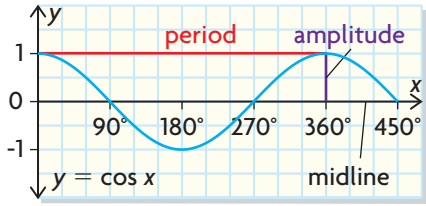
Key Ideas

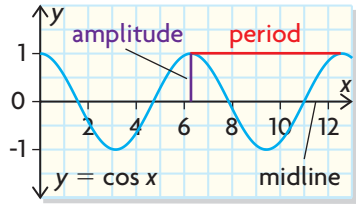
- The function $y = \sin x$ is a periodic function.





- The function $y = \cos x$ is a periodic function.

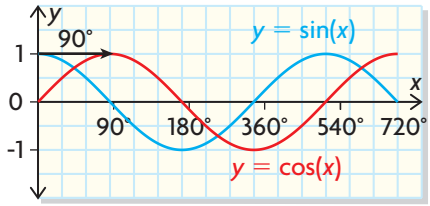


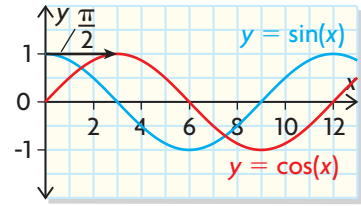


- The graphs of $y = \sin x$ and $y = \cos x$ have the following common characteristics:
  - multiple x-intercepts
  - one y-intercept
  - a domain of $\{x \mid x \in \mathrm{R}\}$
  - a range of $\{y \mid -1 \leq y \leq 1, y \in \mathrm{R}\}$
  - an amplitude of 1
  - a period of 360° or 2π
  - a midline defined by the equation $y = 0$

Need to Know

- The graphs of $y = \sin x$ and $y = \cos x$ are congruent curves.





- The midline of the curves, $y = 0$, is the horizontal line halfway between the maximum and minimum values. The two graphs oscillate about this line.
- The period of a graph is the length of one complete cycle.

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Communication Tip

Graphing technology can graph periodic functions in both degree and radian mode. Make sure your technology is in the desired mode before you enter the function and graph it.

FURTHER Your Understanding

1. a) Using graphing technology and degree measure, graph

$$y = \sin x \text{ and}$$
$$y = \cos x$$

on the same axes, for the domain $\{x \mid 0° \leq x \leq 720°, x \in \mathrm{R}\}$.

b) What is the value of $\cos x$ when the value of $\sin x$ is a maximum? When is the value of $\cos x$ a minimum?

c) What is the value of $\sin x$ when the value of $\cos x$ is a maximum? When is the value of $\sin x$ a minimum?

d) Repeat part a) in radian measure for the domain $\{x \mid 0 \leq x \leq 4\pi, x \in \mathrm{R}\}$.

2. The period of the function

$$y = \sin x$$

is 360° or 2π. Explain why.

3. Compare the graph of the function

$$y = \sin x$$

from 0° to 180° and from 180° to 360°. Is the graph symmetrical? If so, in what way?

4. Compare the graph of the function

$$y = \cos x$$

from 90° to 270° and from 270° to 450°. Is the graph symmetrical? If so, in what way?

5. What is the y-intercept of the graph of each function?

a) $y = \sin x$

b) $y = \cos x$

6. Determine the x-intercepts of each graph over the interval from 0° to 720°.

a) $y = \sin x$

b) $y = \cos x$

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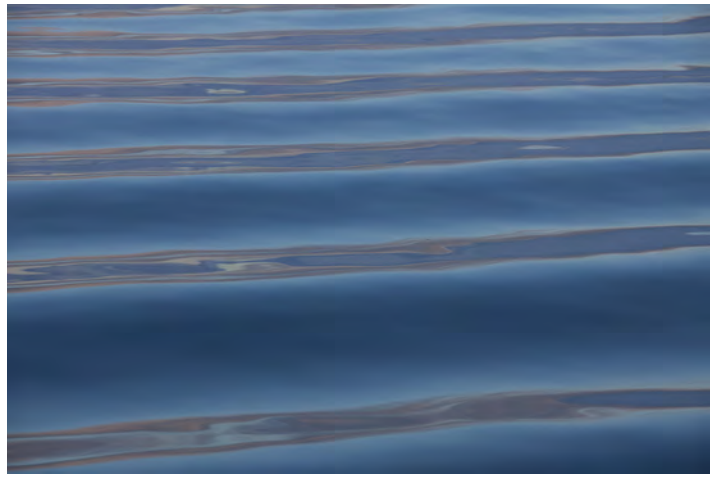
7. The graph of

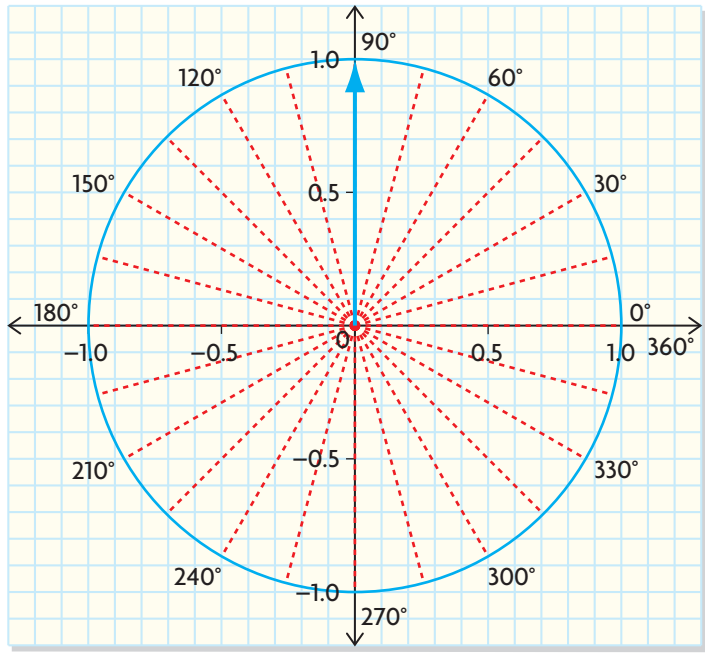
$$y = \sin x$$

is sometimes called a sine wave. Discuss with a partner why this description is appropriate.



8. Adriana says that a sine graph is always a cosine graph translated left by 90°. Do you agree? Explain.

9. Suppose that Carissa put the pointer at the top of the circle and spun it counterclockwise. What would the graph representing the distance from the tip of the pointer to the horizontal axis look like? Explain.



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History | Connection

Not as Easy as π!

How can you create a square with the same area as a given circle using only a straightedge and a compass? In 1897, Edwin Goodwin thought that he had solved this ancient math problem. He showed his solution to representatives of the government of Indiana, and they prepared to enact the result of his solution as law. However, his solution worked only if "the ratio of the diameter (of a circle) and circumference is as five-fourths to four"; that is, if $\pi = 3.2$! This value of π contradicted many known estimates of the value of π.

The ancient Egyptians thought that π was about $\left(\frac{16}{9}\right)^2$ or 3.16. The ancient Babylonians thought that π was about $3\frac{1}{8}$ or 3.125. In Greece, approximately 2300 years ago, Archimedes proved that π was less than $3\frac{10}{70}$ and greater than $3\frac{10}{71}$, or about 3.141 85. Around 480 CE, the Chinese mathematician Zu Chongshi calculated π to be about $\frac{355}{113}$ or 3.141 592 6. With time, people developed more advanced ways to calculate this elusive value. For example:

$$\pi = \frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \ldots$$

Today, computers have calculated the value of π to over 5 trillion decimal places. To 50 decimal places, π is approximately

3.141 592 653 589 793 238 462 643 383 279 502 884 197 169 399 375 10

Fortunately for the government of Indiana, a mathematician named Clarence Waldo noticed the error. The infamous Indiana Pi Bill was not passed.

Clarence Waldo, 1852–1926

A. Since π is not a terminating decimal, why do you think people continue to look for more precise values?

B. The Keystone Pipeline transports crude oil from Hardisty, Alberta, to refineries in the United States. The pipeline is 76.2 cm in diameter and 3456 km long.

i) What is the total surface area of the pipeline?

ii) Suppose that the pipeline company had used Goodwin's value of π to determine the total surface area of steel required to build the pipeline. How much waste would there have been?

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8.3 The Graphs of Sinusoidal Functions

GOAL

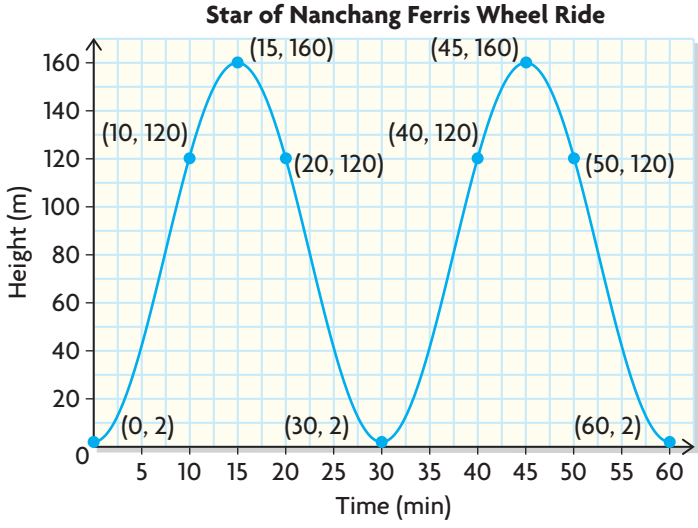
Identify characteristics of the graphs of sinusoidal functions.

YOU WILL NEED

- ruler
- graph paper
- graphing technology

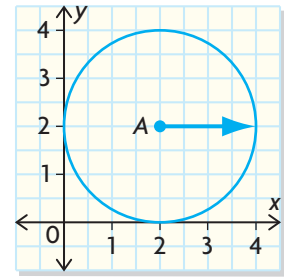
INVESTIGATE the Math

Students in Simone's graduating class went on an exchange trip to China. While they were there, they rode the Star of Nanchang, one of the tallest Ferris wheels in the world. Simone graphed the **sinusoidal function** that represented her ride.



EXPLORE...

- Sketch the graph of the vertical position of the tip of this pointer as a function of degrees of rotation as the pointer spins clockwise for two rotations. How does your graph compare with the graph of the spinner in Lesson 8.2?



sinusoidal function

Any periodic function whose graph has the same shape as that of $y = \sin x$.

? How can you describe Simone's ride using the graph?

A. How can you tell, from Simone's graph, that the lowest part of the Ferris wheel is 2 m off the ground?

B. Determine the maximum value of the graph. What is the height of the Ferris wheel?

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C. Determine the range of the graph. What does this value represent?

D. Determine the amplitude of the graph. What does this value represent?

E. Determine the equation of the midline. What does this value represent?

F. Determine the period of the graph. Explain your method.

G. What length of time is needed for the Star of Nanchang to make one full revolution?

H. How long does it take to get to the top of the Ferris wheel from the bottom?

Reflecting

I. Suppose that a circle of lights were placed on the Star of Nanchang as shown by the red ring, 26 m from the circumference. Sketch Simone's graph on a grid. On the same grid, sketch the graph of the movement of one of the lights. How are the graphs the same? How are they different?

J. George Ferris built the first Ferris wheel in 1893. It was 80.4 m high, and it took 9 min to make one complete revolution. Assume that the minimum height of this Ferris wheel was 0 m. Graph the movement of one of its passenger cars. Compare your graph with Simone's graph. How are the graphs the same, and how are they different?

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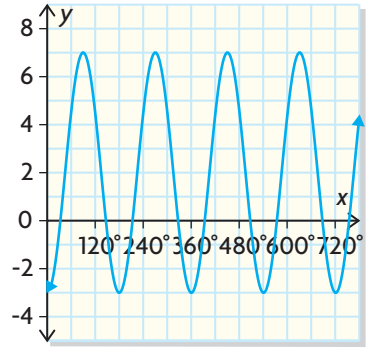
APPLY the Math

EXAMPLE 1 Describing the graph of a sinusoidal function in degree measure

The graph of a sinusoidal function is shown. Describe this graph by determining its range, the equation of its midline, its amplitude, and its period.



Terry's Solution

Range:
Minimum value $= -3$
Maximum value $= 7$
The range of the graph is
$\{y \mid -3 \leq y \leq 7, y \in \mathrm{R}\}$.

I located the minimum and maximum values of the graph to help me determine the range.

I wrote the range using these values.

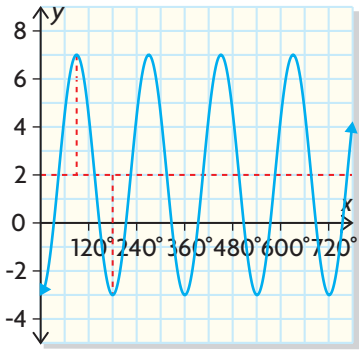
Equation of the midline:

$$y = \frac{\text{maximum value} + \text{minimum value}}{2}$$

$$y = \frac{7 + (-3)}{2}$$

$$y = 2$$

I knew that the midline is the horizontal line halfway between the minimum value and the maximum value.



I verified my solution by looking at the graph. The graph goes 5 units above this horizontal line and 5 units below it.

Amplitude:
Amplitude $= 7 - 2$
Amplitude $= 5$
The amplitude is 5 units.

The amplitude is the vertical distance between the maximum value and the equation of the midline.

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Period:
There is a minimum value at 180°.
The next minimum value is at 360°.
Period = 360° − 180°
Period = 180°
The graph goes through one complete cycle every 180°.

To determine the period, I chose two consecutive minimum points. The difference between the *x*-values of these points is the period.

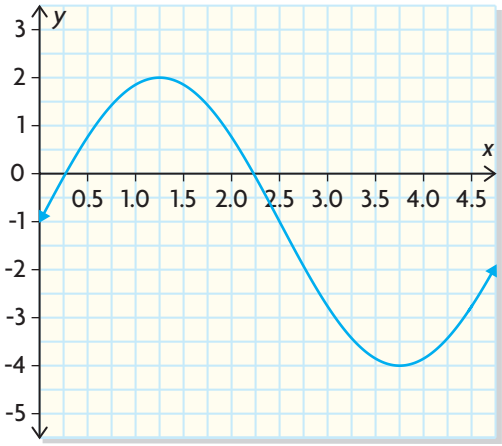
Your Turn

Sketch the graph of a sinusoidal function with the following characteristics:

- The domain is $\{x \mid 0° \leq x \leq 360°, x \in \mathrm{R}\}$.
- The range is $\{y \mid -5 \leq y \leq 9, y \in \mathrm{R}\}$.
- The period is 120°.
- The *y*-intercept is 2.

EXAMPLE 2 Describing the graph of a sinusoidal function in radian measure

The graph of a sinusoidal function is shown. Describe this graph by determining its range, the equation of its midline, its amplitude, and its period.



Bonnie's Solution

Range:
Minimum value = −4
Maximum value = 2

I located the minimum and maximum values of the graph.

The range of the graph is $\{y \mid -4 \leq y \leq 2, y \in \mathrm{R}\}$.

I wrote the range using these values.

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Equation of the midline:

$$y = \frac{\text{maximum value} + \text{minimum value}}{2}$$

$$y = \frac{2 + (-4)}{2}$$

$$y = -1$$

The midline is the horizontal line halfway between the minimum and maximum values.

I verified my solution by looking at the graph. The graph goes 3 units above this line and 3 units below it.

Amplitude:

Amplitude $= 2 - (-1)$

Amplitude $= 3$

The amplitude is 3 units.

The amplitude is the vertical distance between the maximum value and the equation of the midline.

Period:

The maximum value is at $x = 1.25$.

The minimum value is at $x = 3.75$.

Period $= 2(3.75 - 1.25)$

Period $= 2(2.5)$

Period $= 5$

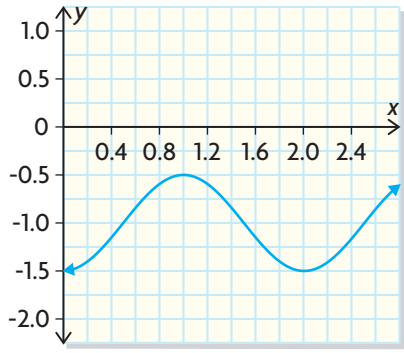
The graph goes through one complete cycle every 5 radians.

Since the graph does not show more than one period, I decided to determine the period by using the maximum point and the minimum point.

The horizontal distance between these two points represents half of a period, so the period is twice this distance.

Your Turn

Determine the range, amplitude, period, and equation of the midline of this sinusoidal function.



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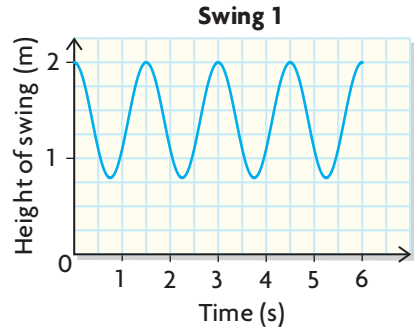
EXAMPLE 3 Connecting a sinusoidal function to oscillating motion

For a physics project, Morgan and Lily had to graph and analyze an example of simple harmonic motion. Morgan swung on a swing, and Lily used a motion detector to measure Morgan's height above the ground over time, as she swung back and forth. The girls then graphed their data as shown. At the end of each cycle, the swing returned to its initial position, which resulted in a sinusoidal graph.



a) Interpret the graph.
b) Determine Morgan's height above the ground at 4 s.

Lily's Solution

a) Range:
Maximum value $= 2.0$
Minimum value $= 0.8$
The range of the height of the swing is $\{y \mid 0.8 \leq y \leq 2.0, y \in \mathrm{R}\}$.

I knew the maximum and minimum values from the data I collected using the motion detector. These values match the maximum and minimum values on the graph.

Equation of the midline:

$$y = \frac{\text{maximum value} + \text{minimum value}}{2}$$

$$y = \frac{2.0 + 0.8}{2}$$

$$y = 1.4$$

I determined the equation of the midline using the range.

I checked my answer by placing my ruler along the line $y = 1.4$ on the graph. Half of the graph is above this line, and half of the graph is below it.

Amplitude:

$$\text{Amplitude} = \frac{\text{maximum value} - \text{minimum value}}{2}$$

$$\text{Amplitude} = \frac{2.0 - 0.8}{2}$$

$$\text{Amplitude} = 0.6$$

The amplitude is half of the vertical distance between the minimum value and the maximum value.

The graph begins at a maximum $(t = 0)$ and reaches the next maximum at $t = 1.5$, so the period is 1.5 s.
This is the length of time for one complete swing, either forward or backward.

I identified two consecutive maximum points and determined the horizontal distance between them. This matches the time that I determined from the motion detector.

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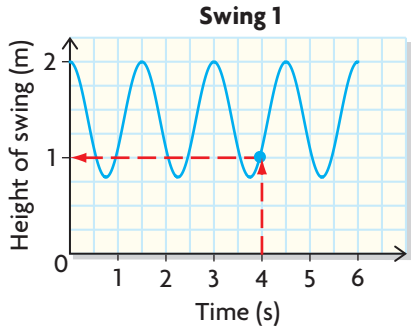
The graph of our simple harmonic model is a sinusoidal function.

- The period of one swing is 1.5 s.
- Morgan swung to a maximum height of 2.0 m.
- The swing is at its minimum height, 0.8 m, each time it passes its position at rest.
- The equation of the midline is $y = 1.4$.
- The amplitude of the swing is 0.6 m.

I summarized our findings for our report.

b)



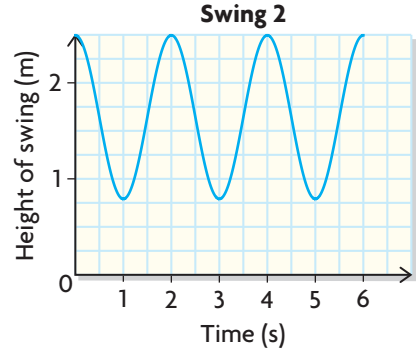
I examined the graph. I drew a vertical line from the horizontal axis, at $t = 4$ s, to a point on the graph. To determine the height, I drew a horizontal line from this point to the vertical axis.

Interpolating from the graph, at $t = 4$ s, Morgan's height is 1 m.

Your Turn

a) To collect a different data set, Lily swung on a different swing, and Morgan created the graph to the right from the data she collected. Interpret this graph.

b) Compare the two swings.



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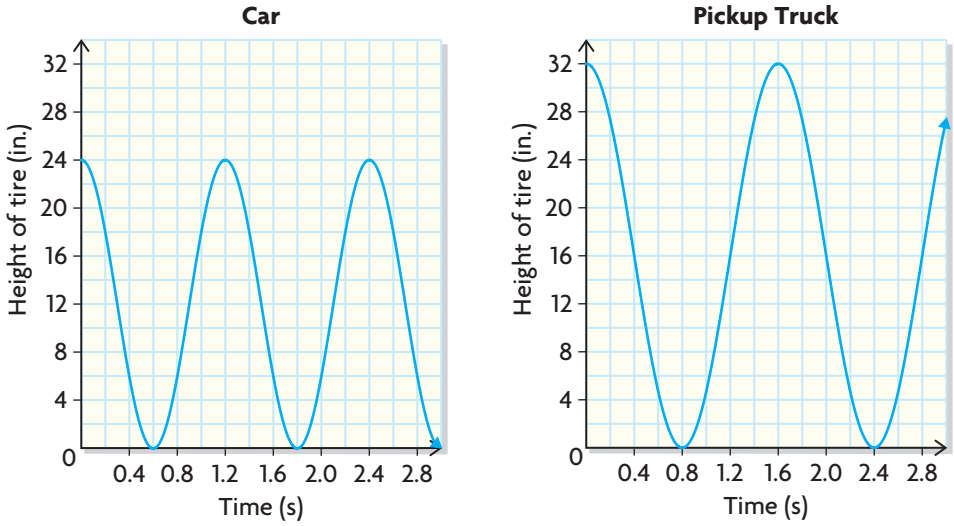
EXAMPLE 4 Comparing two sinusoidal functions

Alexis and Colin own a car and a pickup truck. They noticed that the odometers of the two vehicles gave different values for the same distance. As part of their investigation into the cause, they put a chalk mark on the outer edge of a tire on each vehicle. The following graphs show the height of the tires as they rotated while the vehicles were driven at the same slow, constant speed. What can you determine about the characteristics of the tires from these graphs?



Alexis's Solution

The minimum value of both graphs is 0. This makes sense, because the tires move along the ground.

I examined both graphs. The minimum value is on the horizontal axis.

Maximum values:
Maximum value for car $= 24$ in.
Maximum value for truck $= 32$ in.
These values represent the maximum height of the chalk mark, which is the diameter of each tire.
The diameter of the truck tire is 8 in. greater than the diameter of the car tire.

I compared the maximum values and interpreted what they meant.

Midline of car graph:

$$y = \frac{0 + 24}{2}$$

$$y = 12$$

Midline of truck graph:

$$y = \frac{0 + 32}{2}$$

$$y = 16$$

I determined the midline of each graph.

The centre of the car tire is 12 in. above the ground.
The centre of the truck tire is 16 in. above the ground.

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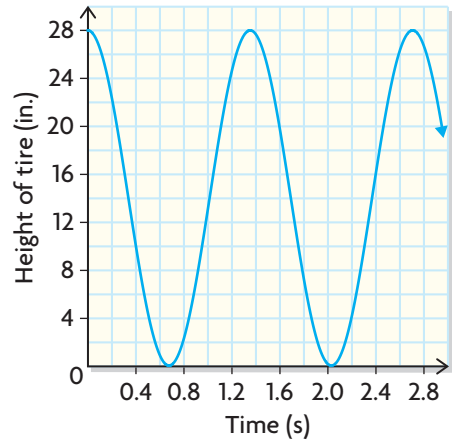
This makes sense, because the axle would be at the centre of each tire, and the height of the centre of each tire above the ground should equal the radius of the tire.

Periods:
Period of car graph = 1.2 s
Period of truck graph = 1.6 s
Since the truck graph has a greater period, the truck tire takes longer to make one rotation.
The truck travels farther in one rotation of the wheels than the car.

I determined the period of each graph.



Your Turn

Alexis installed tires with a larger diameter on the car. She obtained this graph as she tracked the vertical position of a chalk mark.

a) Compare this graph with the original graph for the car tire.

b) A speedometer and an odometer operate based on the number of revolutions that a wheel makes. Discuss, as a class, how larger tires might affect the speedometer and the odometer.

In Summary

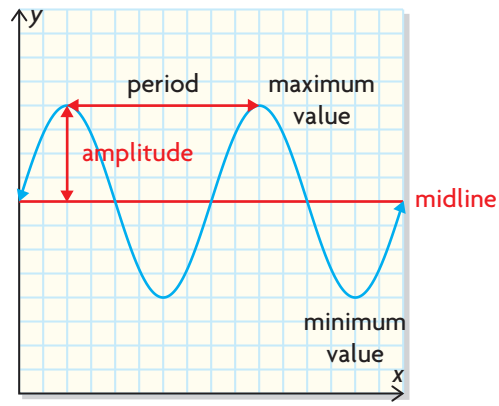
Key Ideas

- Sinusoidal functions can be used as models to solve problems that involve repeating or periodic behaviour.
- Functions whose graphs have the same shape and characteristics as the sine function are called sinusoidal functions.

Need to Know

- You can determine the characteristics of a sinusoidal function from its graph:
  - The **period** is the horizontal distance between consecutive maximum values or consecutive minimum values. It is also twice the horizontal distance between a maximum value and the next minimum value.
  - The **equation of the midline** is the average of the maximum and minimum values:

$$y = \frac{\text{maximum value} + \text{minimum value}}{2}$$



  - The **amplitude** is the positive vertical distance between the midline and either a maximum or minimum value. It is also half of the vertical distance between a maximum value and a minimum value.

$$\text{Amplitude} = \frac{\text{maximum value} - \text{minimum value}}{2}$$

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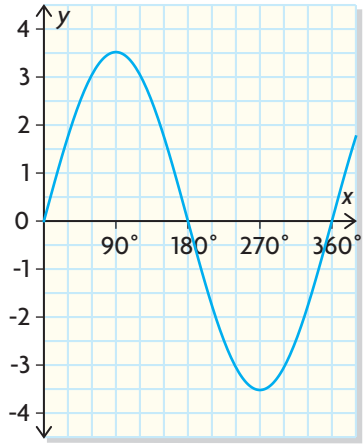
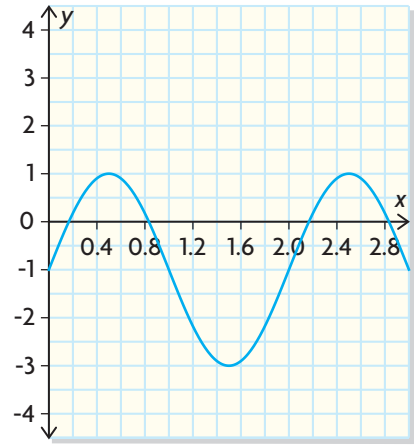
CHECK Your Understanding

1. Determine the range and amplitude of each graph.

a)

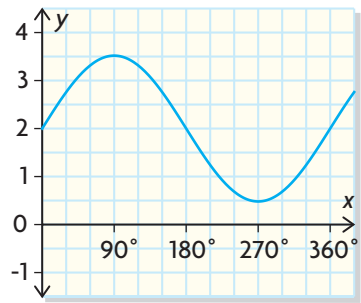


b)



2. Determine the equation of the midline and the amplitude of each graph.

a)



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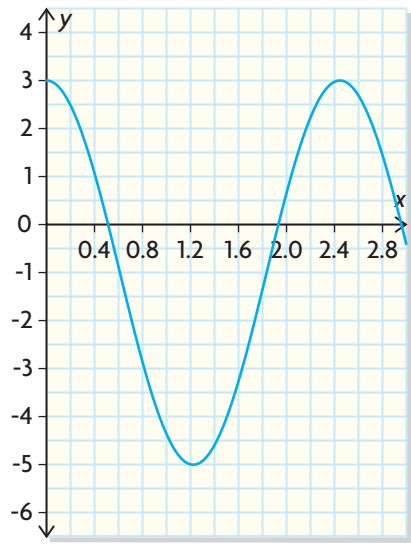

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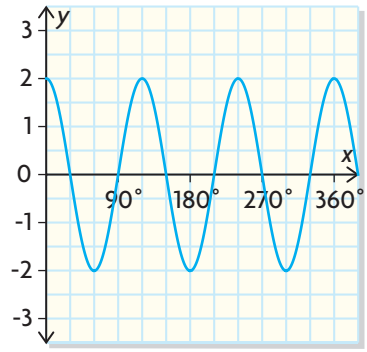
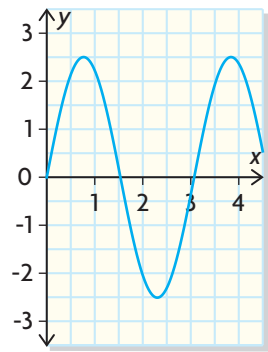
3. Determine the period of each graph.

a)



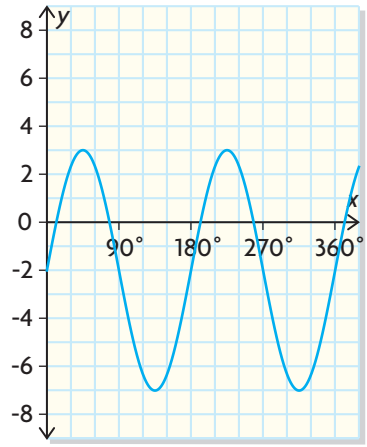
b)



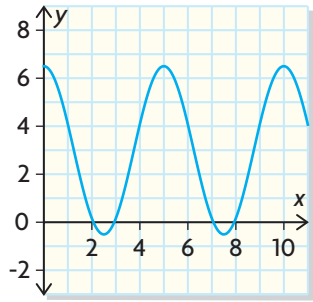
PRACTISING

4. Determine the range, amplitude, equation of the midline, and period of each graph.

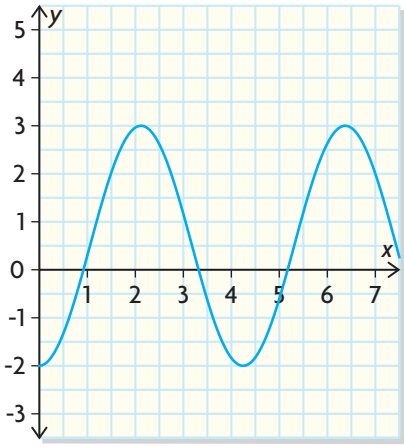
a)



b)



5. Determine the range, amplitude, equation of the midline, and period of the graph below.



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6. Sketch a possible graph of a sinusoidal function with each set of characteristics.

a) Domain: $\{x \mid 0° \leq x \leq 180°, x \in \mathrm{R}\}$
Range: $\{y \mid 2 \leq y \leq 6, y \in \mathrm{R}\}$
Period: 90°
y-intercept: 4

b) Domain: $\{x \mid 0 \leq x \leq 16, x \in \mathrm{R}\}$
Maximum value: 3
Minimum value: -3
Period: 8
y-intercept: -3

7. In 2011, the London Eye was the largest Ferris wheel in the western hemisphere. It rises 135 m above the ground and takes the same amount of time to make one rotation as the Star of Nanchang. How would the graph of a passenger's ride on the London Eye be the same as Simone's graph? How would it be different?

The London Eye was completed in 1999.

The West Edmonton Mall, which opened in 1981, was the largest mall in North America in 2011.

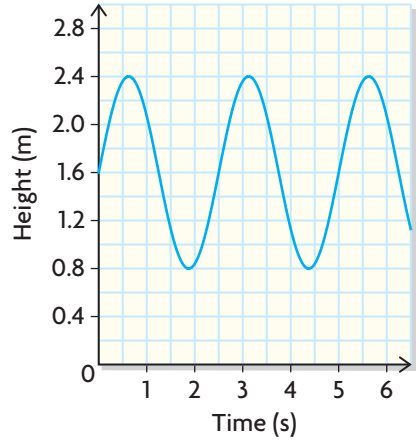
8. Mussab is sitting in an inner tube in the wave pool at West Edmonton Mall. The depth of the water below him, in terms of time, during a series of waves can be represented by the graph shown.

a) What is the depth of the water below Mussab when no waves are being generated?

b) How high is each wave?

c) How long does it take for one complete wave to pass?

d) What is the approximate depth of the water below Mussab after 4 s? What is the depth of the water below Mussab at 7.5 s? Assume that the waves continue at the same rate.



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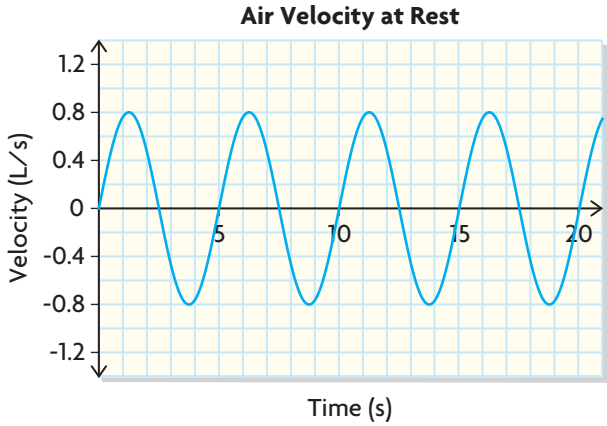
9. When you breathe, the air entering your lungs has a positive velocity and the air exiting your lungs has a negative velocity. The relationship between velocity, in litres of air per second (L/s), and time, in seconds, for an adult at rest, can be modelled by the graph shown.

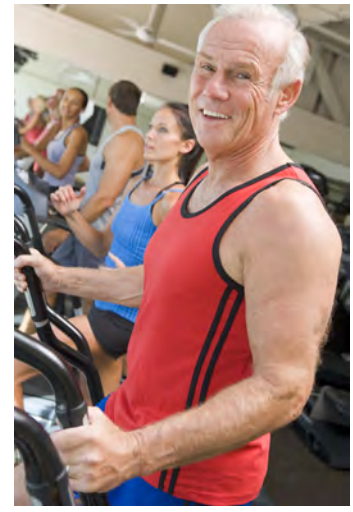
a) What is the equation of the midline? What does it represent in this situation?

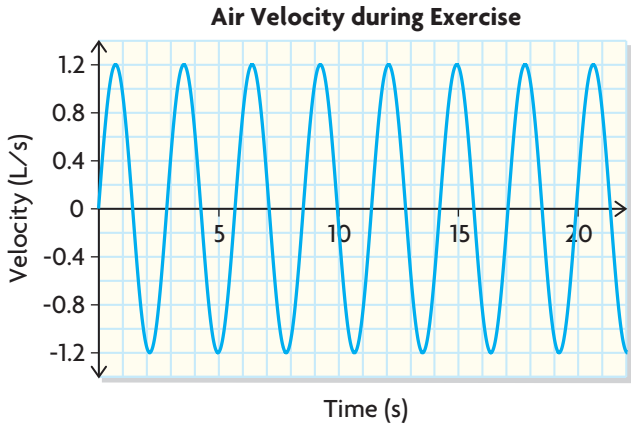
b) What is the amplitude of the function?

c) What is the period of the function? What does it represent in this situation?



10. When you exercise, the velocity of the air entering and exiting your lungs, measured in litres per second, changes in terms of time, measured in seconds. The following graph models the relationship between velocity and time for an adult who is exercising.





a) According to this model, does an adult take more breaths per minute when exercising, or just deeper breaths, than an adult at rest (modelled in question 9)? How do you know?

b) What characteristic (period, equation of midline, or amplitude) of this graph has changed, compared with the graph in question 9?

c) What is the maximum velocity of the air entering the lungs? Include the appropriate units of measure.

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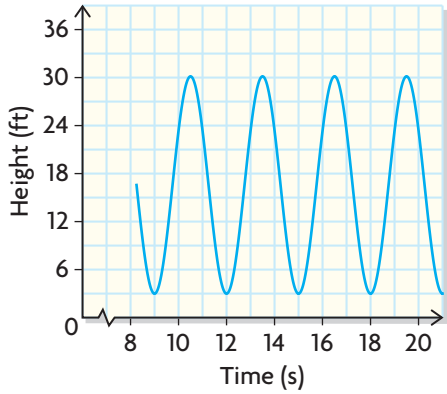
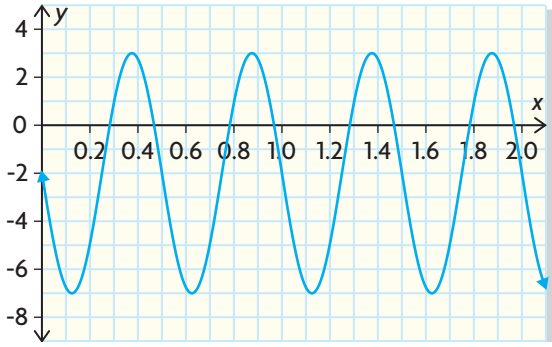
Karen Cockburn is the only trampoline athlete to have won a medal at three consecutive Olympic Games: 2000, 2004, and 2008.

11. A competitive gymnast's coach graphs one particular series of jumps. Describe the gymnast's jumps using the graph.



12. Determine the characteristics of this graph.

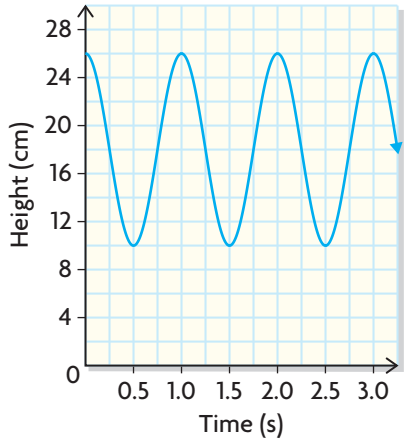


13. Caitlin and Rahim conducted an experiment in physics class. They swung two different pendulums above a table and recorded the motion of the pendulums in graphs.

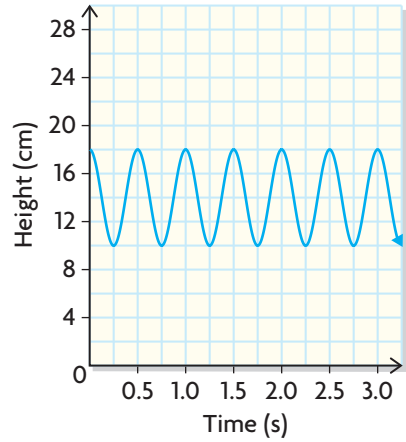
a) Compare the periods, minimum values, maximum values, and amplitudes of the two pendulums.

b) Which pendulum is longer? Explain.

Pendulum A



Pendulum B



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14. Takoda and his sister Talula are hoop dancers. In part of a dance, they spin hoops about their arms. Each of the following graphs indicates the height of a point on a hoop, measured from the ground, that Takoda or Talula is spinning vertically.

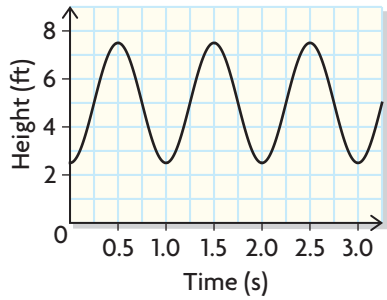
a) What does the amplitude represent?

b) Which hoop is the smallest?

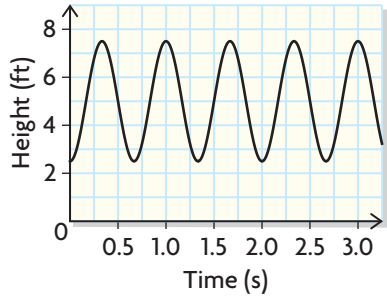
c) Which hoop is being spun at the slowest rate? Which is being spun at the fastest rate?

d) Which hoop do you think the shorter person is spinning? Explain.

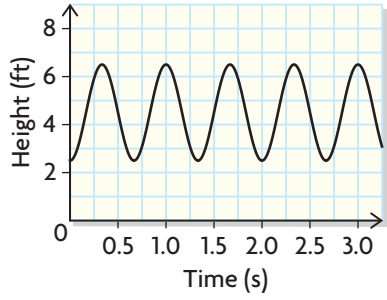
Hoop 1

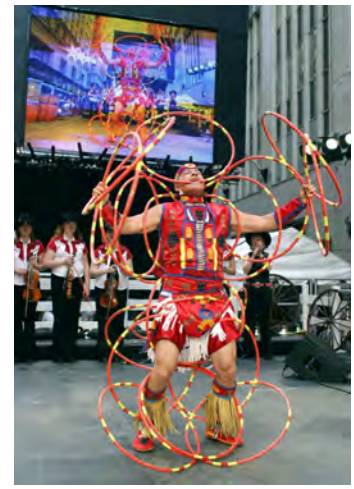


Hoop 2



Hoop 3





Hoop dancers form shapes using hoops to tell stories in their dances. The hoop dance, or some form of it, has existed among First Nations across North America for hundreds of years. The modern form of hoop dancing came into being in the 1930s. Hoop dance competitions are often part of Pow Wows. Modern hoop dances contain spectacular moves and shapes, and are performed by both men and women.

Closing

15. How can you determine the amplitude, range, and equation of the midline of a given sinusoidal graph?

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Extending

16. In high winds, the top of a flagpole sways back and forth. The distance that the tip of the flagpole vibrates away from its resting position can be modelled by the function

$$d(t) = 2 \sin (4\pi t)$$

where $d(t)$ represents the distance in centimetres and t represents the time in seconds. If the wind speed decreases by 20 km/h, the distance can be modelled by the following function:

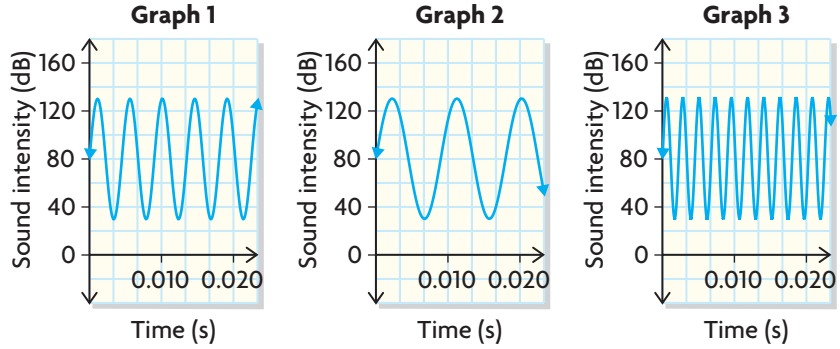
$$d(t) = 1.5 \sin (4\pi t)$$

Plot the two graphs using technology in radian mode. How are the period, midline, and amplitude affected when the wind speed decreases?

17. Music is composed of sound waves, which can be modelled using sinusoidal functions. For example, these graphs show three consecutive A pitches, but not in increasing order, for $\{x \mid 0 \leq x \leq 0.02 \text{ s}, x \in \text{R}\}$. A higher **frequency** is equivalent to a higher pitch.

frequency

The number of times that a cycle occurs in a given time period. For example, the fourth A note on a piano has a frequency of 440 Hz or 440 cycles per second.



a) Arrange the graphs in order, from the lowest pitch to the highest pitch.

b) Compare the graphs. What do you notice about the frequencies of consecutive A pitches?

c) Graph 1 has a frequency of 220 Hz, or 220 cycles per second. What is the frequency of each of the other two graphs?

d) Research the sound waves that are produced by musical instruments. For example, you could consider the following questions:

- Are all sound waves sinusoidal?
- What impact does the size of an instrument have on its sound waves?
- What other elements have an impact on its sound waves? Give some examples.

Communication | *Tip*

A millisecond is 0.001 s.

18. Electrical outlets provide power using an alternating current (AC) that alternates according to a sinusoidal function. The typical voltage in Canada oscillates between −170 V (volts) and +170 V, with a frequency of 60 Hz (hertz), or 60 times per second.

a) How many milliseconds does it take for one complete cycle of electricity?

b) Draw a graph that represents the alternating current for three complete cycles. Label all the axes.

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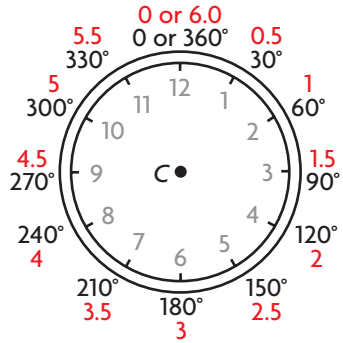
8 Mid-Chapter Review

FREQUENTLY ASKED Questions

Q: How can you estimate the measure of an angle given in radians using degree measure, and vice versa?

A: Use benchmarks. Since 1 radian is equivalent to approximately 57.3°, use a benchmark that 60° is slightly more than 1 radian. Use this to determine that a benchmark for 30° is slightly more than 0.5 in radian measure. To get a rough estimate, use approximations of 2 radians for 120°, 3 radians for 180°, and so on.

For example, an angle of 210°, or 180° + 30°, is equivalent to about 3.5 in radian measure. This estimate is slightly low.



Rough benchmarks

When estimating the measure of an angle given in degree measure, recall that 180° is equivalent to π in radian measure, or about 3.14. Similarly, 360° is equivalent to 2π, or about 6.28. This means that 90° is equivalent to slightly more than 1.5 in radian measure, or approximately 1.6.

For example, an angle given as 4.8, or 3.2 + 1.6, is equivalent to about 180° + 90°, or about 270°.

Benchmarks can also be used to estimate large angles.

For example, benchmarks can be used to estimate 585° in radian measure.

585° = 360° + 180° + 45°
This measure is equivalent to

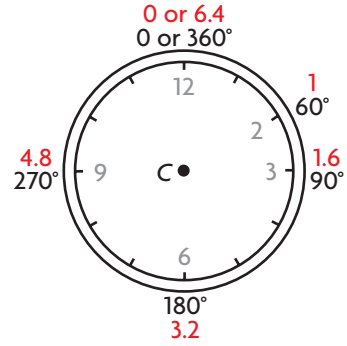
6.4 + 3.2 + 0.8

360° is equivalent to about 6.4.
180° is equivalent to about 3.2.
45°, which is half of 90°, is equivalent to about 0.8.

A reasonable estimate of 585° in radian measure is 10.4.

Study *Aid*

- See Lesson 8.1, Examples 2 to 4.
- Try Mid-Chapter Review Questions 1 and 2.



These benchmarks are a little more precise.

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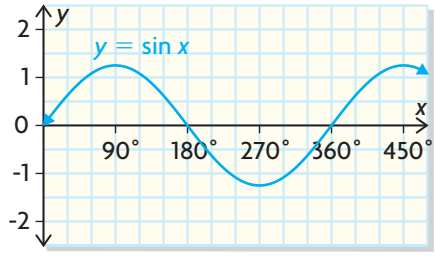
Study Aid

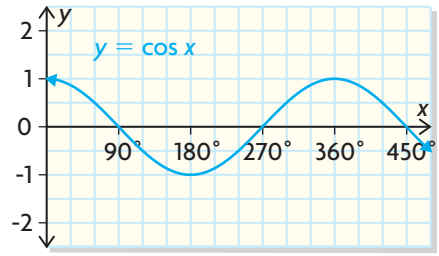
- See Lesson 8.2.
- Try Mid-Chapter Review Questions 4 and 5.

Q: How are the graphs of a sine function and a cosine function the same? How are they different?

A: Same: They have the same appearance. They have the same amplitude, midline, period, and maximum and minimum values.

Different: The graph of $y = \sin x$ has a y-intercept at the origin; the graph of $y = \cos x$ has a y-intercept of 1. The graph of $y = \cos x$ can be thought of as the graph of $y = \sin x$, but translated to the left by 90°.





Study Aid

- See Lesson 8.3, Examples 1 and 2.
- Try Mid-Chapter Review Questions 6 and 7.

Q: How can you determine the characteristics of a sinusoidal function from its graph?

A:

- The amplitude is half of the vertical distance between the maximum and minimum values:

$$\frac{\text{max} - \text{min}}{2}$$

- The equation of the midline is

$$y = \frac{\text{max} + \text{min}}{2}$$

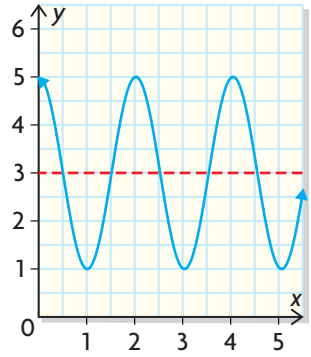
- The period is the horizontal distance between any two consecutive maximum points or any two consecutive minimum points.

For example, this graph has a minimum value of 1 and a maximum value of 5, so its range is $\{y \mid 1 \leq y \leq 5, y \in \text{R}\}$.

Its amplitude is $\frac{5 - 1}{2}$ or 2.

The equation of its midline is $y = \frac{5 + 1}{2}$ or $y = 3$.

There are consecutive minimum values at $x = 1$ and $x = 3$, so its period is $3 - 1$, or 2.



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PRACTISING

Lesson 8.1

1. Estimate, to the nearest tenth, the measure of each angle in radians.
a) 135° **b)** 210° **c)** 400° **d)** 515°

2. Estimate, to the nearest degree, the measure of each angle in degrees.
a) 1.1 **b)** 3.2 **c)** 6.5 **d)** 9.5

3. Arrange the following angle measures in descending order:
π, 200°, 1.5, 190°, 310°, 0.5, 20°, 0.1, 400°

Lesson 8.2

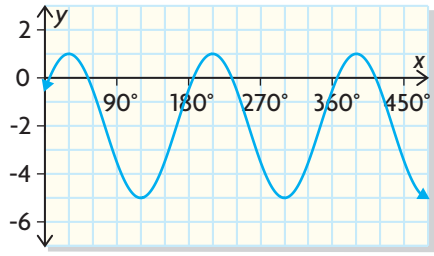
4. **a)** State the range, amplitude, period, and equation of the midline of the graph of $y = \sin x$.
b) How are the graphs of the functions
$y = \cos x$ and $y = \sin x$
the same? How are the graphs different?

5. Consider the function $y = \sin x$ for the domain $\{x \mid 0° \leq x \leq 720°, x \in \mathrm{R}\}$.
a) At what values of x is the function equal to zero?
b) At what values of x is the function at a maximum? At a minimum?

Lesson 8.3

6. State the range, amplitude, period, and equation of the midline of each graph.

a)



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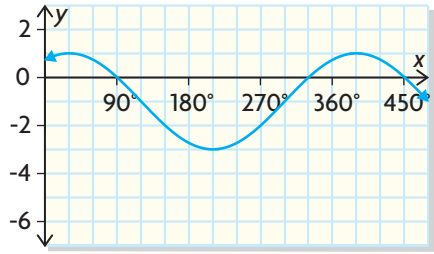
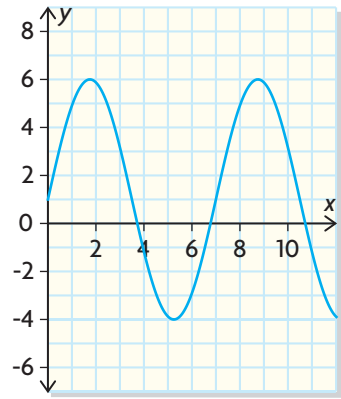


7. Determine the characteristics of each graph.

a)



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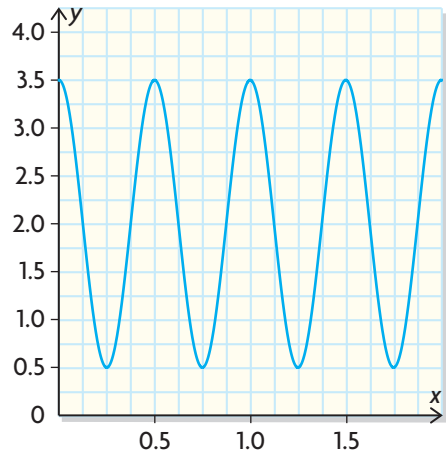
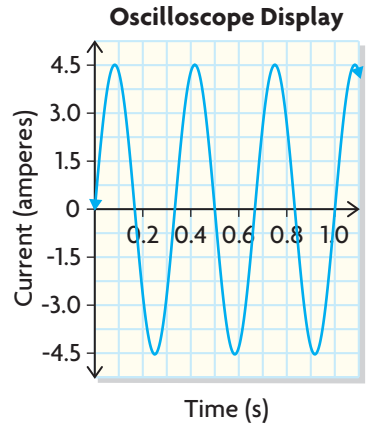


8. An oscilloscope in an alternating current circuit shows a sine curve on its display.



a) What is the period of the function?
b) What is the equation of the midline of the function?
c) What is the amplitude of the function?

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8.4 The Equations of Sinusoidal Functions

YOU WILL NEED

- graphing technology

EXPLORE...

- Determine the maximum value, minimum value, and range of

$$y = 3 \sin x + 2$$

GOAL

Identify characteristics of the equations of sinusoidal functions.

INVESTIGATE the Math

Cesar and Malachi have to do a project on local Winnipeg history. They have decided to research Grant's Old Mill, located at the Red River Settlement. Their math teacher said that they could describe the motion of a point on the rim of the water wheel using the sinusoidal function

$$y = a \sin b(x - c) + d$$

where a, b, c, and d are real numbers.

Métis leader Cuthbert Grant built this water wheel in 1829. It was the first source of hydro power in Manitoba.

? How does changing the values of a, b, c, and d in a sinusoidal function affect the graph of the function?

A. Graph, in radian mode,

$$y = \sin x$$

for the domain $\{x \mid 0 \leq x \leq 4\pi, x \in \mathrm{R}\}$. Then graph the following functions on the same axes.

i) $y = 3 \sin x$ **ii)** $y = 5 \sin x$ **iii)** $y = \frac{1}{2} \sin x$

B. Compare your graphs in part A.

i) How does changing the value of a affect the graph of $y = \sin x$? Show your answer in a sketch.

ii) What does the value of a tell you about a water wheel? Explain.

C. On a new grid, graph, in radian mode,

$$y = \sin x$$

for the domain $\{x \mid 0 \leq x \leq 4\pi, x \in \mathrm{R}\}$. Then graph these functions on the same axes.

i) $y = \sin 2x$ **ii)** $y = \sin 3x$ **iii)** $y = \sin \frac{1}{2}x$

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D. Compare your graphs in part C.

i) How does changing the value of b affect the graph of $y = \sin x$? Show your answer in a sketch.

ii) What does the value of b tell you about a water wheel? Explain.

E. On a new grid, graph, in radian mode,

$$y = \sin x$$

for the domain $\{x \mid 0 \leq x \leq 4\pi, x \in \mathrm{R}\}$. Graph the following functions on the same axes.

i) $y = \sin (x - 1.6)$ **ii)** $y = \sin (x - 3.1)$ **iii)** $y = \sin (x + 0.8)$

F. Compare your graphs in part E.

i) How does changing the value of c affect the graph of $y = \sin x$? Show your answer in a sketch.

ii) What does the value of c tell you about a water wheel? Explain.

G. On a new grid, graph, in radian mode,

$$y = \sin x$$

for the domain $\{x \mid 0 \leq x \leq 4\pi, x \in \mathrm{R}\}$. Graph these functions on the same axes.

i) $y = 4 \sin x + 3$ **ii)** $y = 4 \sin x - 1$ **iii)** $y = 4 \sin x + 4$

H. Compare your graphs in part G.

i) How does changing the value of d affect the graph of $y = \sin x$? Show your answer in a sketch.

ii) What does the value of d tell you about a water wheel? Explain.

Communication *Tip*

The expression $\sin x + c$ indicates that sine operates only on x. To indicate that sine operates on the expression $x + c$, parentheses are used: $\sin (x + c)$.

Reflecting

I. Consider the sinusoidal equation

$$y = a \sin bx + d$$

Explain how to determine each of the following characteristics of the function, based on the values of a, b, and d:

- the amplitude
- the period
- the equation of the midline

J. Consider the equation

$$y = a \sin (x - c) + d$$

How does the value of c affect the graph?

K. Do you think the values of a, b, c, and d will affect a sine graph in the same way if the angles are measured in degrees? Explain.

L. Will the values of a, b, c, and d affect a cosine graph in the same way that they affect a sine graph? Explain.

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APPLY the Math

EXAMPLE 1 Determining the characteristics of a cosine function based on its equation

Consider the function

$$y = 2 \cos 4x + 1$$

for $\{x \mid 0° \leq x \leq 360°, x \in \mathrm{R}\}$.

a) Describe the graph of the function by stating the amplitude, equation of the midline, range, and period, as well as the relevant horizontal translation of $y = \cos x$.

b) Verify your description by drawing a graph of this function using graphing technology.

T.J.'s Solution

a) $y = a \cos b(x - c) + d$
$y = 2 \cos 4x + 1$
$a = 2, b = 4, c = 0, d = 1$

I determined *a*, *b*, *c*, and *d*.

The amplitude of the graph is a, which is 2.
The equation of the midline is $y = d$, or $y = 1$.

I determined the amplitude and the equation of the midline.

Minimum value $= d - a$ Maximum value $= d + a$
Minimum value $= 1 - 2$ Maximum value $= 1 + 2$
Minimum value $= -1$ Maximum value $= 3$

I determined the minimum and maximum values.

The range of the graph is $\{y \mid -1 \leq y \leq 3, y \in \mathrm{R}\}$.

I wrote the range using these values.

Since $b = 4$, the graph completes four cycles in 360°.

$$\text{Period} = \frac{360°}{b}$$

$$\text{Period} = \frac{360°}{4}$$

$$\text{Period} = 90°$$

I thought of *b* as an indicator of speed. The graph completes a cycle four times more quickly than the graph of $y = \cos x$.

Since $c = 0$, the graph has a y-intercept at the maximum value, or $y = 3$.

The graph has not been translated horizontally, since the graph of $y = \cos x$ also has a y-intercept at its maximum value.

I knew that the graph has not been translated horizontally, because $c = 0$.

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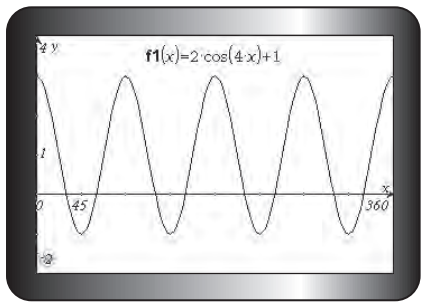
b) $y = 2 \cos 4x + 1$



I examined my graph. The minimum is -1 and the maximum is 3, so the amplitude is 2. There are four complete periods in my window, so I confirmed that the period is 90°.
The graph starts at a maximum. This is a cosine function, so the graph has not been translated horizontally.
My description is correct.

I entered the equation into my graphing calculator. I set the domain as $\{x \mid 0° \leq x \leq 360°, x \in \mathrm{R}\}$ and decided on a scale for x using an interval of 45°, which is half of the period I had determined.

I set the y-axis to show 1 less than the minimum and 1 more than the maximum.

Your Turn

Consider the graph of

$$y = 5 \cos \frac{1}{2}x - 3$$

where x is measured in degrees.

a) Describe the graph of the function by stating the amplitude, equation of the midline, range, and period, as well as the relevant horizontal translation of $y = \cos x$.

b) Verify your description by drawing the graph using graphing technology.

EXAMPLE 2 Determining the characteristics of a sine function based on its equation

Consider the function

$$y = 3 \sin 2(x - 45°)$$

a) Describe the graph of the function by stating the amplitude, equation of the midline, range, and period, as well as the relevant horizontal translation of $y = \sin x$.

b) Verify your description by drawing the graph using graphing technology.

c) How would the graph of $y = 3 \cos 2(x + 45°)$ be the same? How would it be different?

Communication *Tip*

The degree symbol on 45 indicates that the independent variable x, is measured in degrees. If no degree symbol is present in an equation, assume that x is measured in radians, unless stated otherwise.

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Vanessa's Solution

a) $y = a \sin b(x - c) + d$
$y = 3 \sin 2(x - 45°)$
$a = 3, b = 2, c = 45°, d = 0$

I determined a, b, c, and d.

The amplitude of the graph is a, which is 3.

The equation of the midline is $y = d$, or $y = 0$; that is, the x-axis.

I described the graph using these values.

I wrote the range using the minimum and maximum values.

$$\text{Minimum value} = d - a \qquad \text{Maximum value} = d + a$$
$$\text{Minimum value} = 0 - 3 \qquad \text{Maximum value} = 0 + 3$$
$$\text{Minimum value} = -3 \qquad \text{Maximum value} = 3$$

The range of the graph is $\{y \mid -3 \leq y \leq 3, y \in \mathrm{R}\}$.

Since $b = 2$, the graph completes two cycles in 360°.

I determined the period.

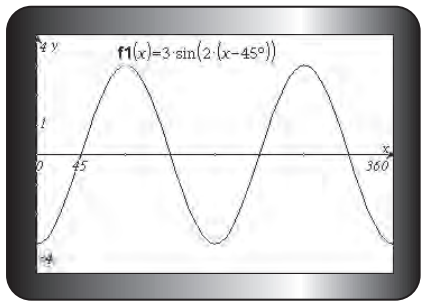
$$\text{Period} = \frac{360°}{b}$$
$$\text{Period} = \frac{360°}{2}$$
$$\text{Period} = 180°$$

The graph has been translated 45° to the right, since $c = 45°$.

I knew that the value of c indicates the horizontal translation.

b) $y = 3 \sin 2(x - 45°)$

I entered the equation into my graphing calculator.
I set the domain as $\{x \mid 0 \leq x \leq 360°\}$.
I set the range to show 1 less than the minimum and 1 more than the maximum.



I examined my graph. The minimum is -3, and the maximum is 3. Therefore, the amplitude is 3.

There are two complete periods in my window, so I confirmed that the period is 180°.

The graph starts at a minimum, so it has been translated horizontally. It first intersects the midline at 45°, so it has been translated horizontally by 45°.

My description is correct.

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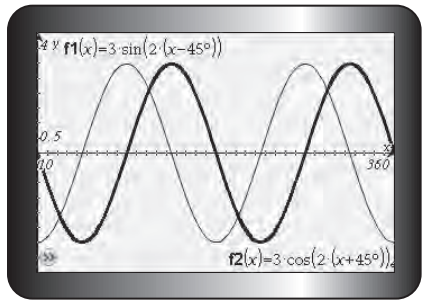
c) The graphs of the functions

$y = 3 \sin 2(x - 45°)$ and
$y = 3 \cos 2(x + 45°)$

should have the same amplitude and period, since a and b are the same.

The graph of the second function is translated by $-45°$, or $45°$ to the left of $y = \cos x$.

Since the value of c is opposite the value of c in the first function, the graph is translated to the left.



I graphed the second equation.

The second graph has been translated 45° to the right, as compared to a standard graph of the cosine function. Otherwise, the two graphs have the same characteristics. My description is correct.

Your Turn

Consider the function

$$y = 4 \cos 3(x - 60°)$$

a) Describe the graph of the function by stating the amplitude, equation of the midline, range, and period, as well as the relevant horizontal translation of $y = \cos x$.

b) Confirm your description by drawing the graph using graphing technology.

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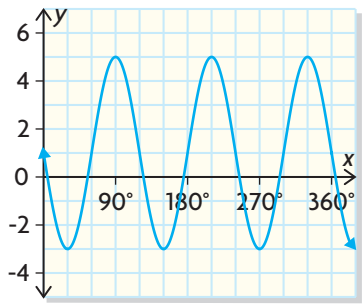
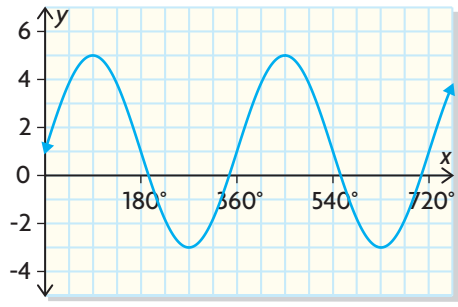
EXAMPLE 3 Matching equations to graphs

Match each graph with the corresponding equation below.

a) **Graph 1**



b) **Graph 2**



i) $y = 4\cos(x - 90°) + 1$

ii) $y = 5\sin 3(x - 60°)$

iii) $y = 4\sin 3(x - 60°) + 1$

iv) $y = 4\cos 3(x - 60°) + 1$

Anya's Solution

a) $\text{Amplitude} = \dfrac{\text{maximum value} - \text{minimum value}}{2}$

$\text{Amplitude} = \dfrac{5 - (-3)}{2}$

$\text{Amplitude} = 4$

The amplitude of equation ii) is 5, so equation ii) is eliminated.

I could not tell if the graph represents a sine function or a cosine function. I had to examine the characteristics of the graph.

I determined the amplitude using values from the graph.

$\text{Period} = 210° - 90°$

$\text{Period} = 120°$

$\text{Period} = \dfrac{360°}{b}$

$120° = \dfrac{360°}{b}$

$b = 3$

There are consecutive maximum points at 90° and 210°. I determined the period of the graph using these points.

I determined the value of b in the equation $y = a\sin b(x - c) + d$.

The value of b in equation i) is 1, so equation i) is eliminated.

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If the graph represents a cosine function, then it has been translated by 90° to the right. Equation iv) represents a cosine function that has been translated 60° to the right, not 90°, so equation iv) is eliminated.

I knew that the graph represents either a cosine function or a sine function. I compared the possible translation of a cosine function to the value of c in the equations.

By elimination, equation iii) must be correct.

Verify:

I verified my deduction.

Equation iii) represents a sine function that has been translated 60° to the right, like the graph. It has an amplitude of 4 and a period of 120°. The value of b is 3, and the equation of the midline is $y = 1$. All of these characteristics match the graph. Equation iii) is the equation of this graph.

b) I know that equation iii) is eliminated, since equation iii) matches graph a).

I could not tell if the graph represents a sine function or a cosine function. I had to examine the characteristics of the graph.
I determined the amplitude of the graph.

$$\text{Amplitude} = \frac{\text{maximum value} - \text{minimum value}}{2}$$

$$\text{Amplitude} = \frac{5 - (-3)}{2}$$

$$\text{Amplitude} = 4$$

The amplitude of equation ii) is 5, so equation ii) is eliminated.

There are consecutive maximum points at 90° and 450°. I determined the period of the graph using these points.

$$\text{Period} = 450° - 90°$$

$$\text{Period} = 360°$$

$$\text{Period} = \frac{360°}{b}$$

I determined the value of b in the equation of the sinusoidal function.

$$360° = \frac{360°}{b}$$

$$b = 1$$

Only equation i) has $b = 1$, so equation i) must be the equation of this graph.

I verified my deduction.

Verify:

Equation i) represents a cosine function that has been translated 90° to the right, like the graph. It has an amplitude of 4 and a period of 360°. The value of b is 1, and the equation of the midline is $y = 1$. All of these characteristics match the graph.

Your Turn

Write another equation that matches each graph in Example 3.

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EXAMPLE 4 Solving a problem using a sinusoidal function

The Far North is called "the Land of the Midnight Sun" for a good reason: during the summer months, in some locations, the Sun can be visible for 24 h a day. The number of hours of daylight in Iqaluit, Nunavut, can be represented by the function

$$y = 8.245 \sin 0.0172(x - 80.988) + 12.585$$

where x is the day number in the year.

a) How many hours of daylight occur in Iqaluit on the following days?
- **i)** the shortest day of the year
- **ii)** the longest day of the year

b) In some years, June 21 is the longest day. Suppose that the Sun were to set in Iqaluit at 11:01 p.m. on June 21. At what time did the Sun rise?

c) What is the period of this sinusoidal function? Explain how the period relates to the context of the problem.

d) What does the value of c, 80.988, represent in the context of the problem?

The Sun rising in Iqaluit on the shortest day of the year

Tyler's Solution: Analyzing the equation

a) $y = a \sin b(x - c) + d$
$y = 8.245 \sin 0.0172(x - 80.988) + 12.585$
$a = 8.245$, $b = 0.0172$, $c = 80.988$, $d = 12.585$

I determined a, b, c, and d.

Minimum value $= d - a$
Minimum value $= 12.585 - 8.245$
Minimum value $= 4.340$
Maximum value $= d + a$
Maximum value $= 12.585 + 8.245$
Maximum value $= 20.830$

I determined the minimum and maximum hours of daylight.

$$0.34 \text{ h} \cdot \frac{60 \text{ min}}{1 \text{ h}} = 20.4 \text{ min}$$

I converted the decimal part of the daylight hours to minutes.

The shortest day of the year has 4.34 h of daylight, or about 4 h 20 min.

$$0.83 \text{ h} \cdot \frac{60 \text{ min}}{1 \text{ h}} = 49.8 \text{ min}$$

The longest day of the year has 20.83 h of daylight, or about 20 h 50 min.

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b) Sunset is at 11:01 p.m.
Sunrise is 20 h 49.8 min earlier.
12 h earlier is 11:01 a.m.

11:01 a.m. less 8 h 49.8 min is about 2:11 a.m.
The Sun rose at 2:11 a.m.

Since the longest day is 20 h 50 min, sunrise must occur this length of time before 11:01 p.m. First, I subtracted 12 h.

I still needed to subtract 8 h 50 min. To make subtraction easier, I thought of 11:01 a.m. as being 10 h and 61 min. I subtracted 8 h 49.8 min.

c) $\text{Period} = \frac{2\pi}{b}$

$\text{Period} = \frac{2\pi}{0.0172}$

$\text{Period} = 365.301...$

I calculated the period. Since no units are indicated, I used radian measure.

The period for $y = \sin x$ is 2π.

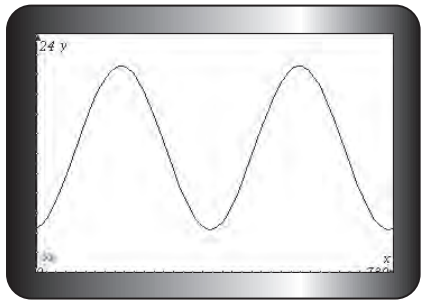
Earth revolves about the Sun about every 365.25 days, so the period and the length of one rotation are about the same.

d) The value of c is 80.988, or about 81 days.
The 81st day of the year is March 22, which is around the first day of spring.

I knew that c represents the horizontal translation of the graph of $y = \sin x$. Since this is the equation of a sine function, the graph crosses the midline at day 81 and is increasing.

Siddiqa's Solution: Analyzing the graph of the equation using technology

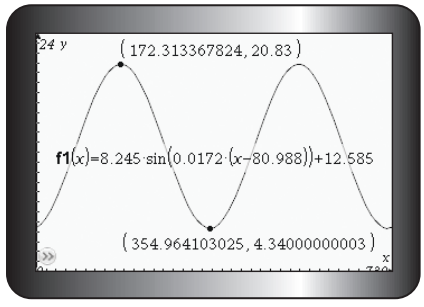
a)



I entered the equation on my graphing calculator.
I used a domain of 2 years:
$\{x \mid 0 \leq x \leq 730, x \in \mathrm{R}\}$

I used a range of 1 day, or 24 h:
$\{y \mid 0 \leq y \leq 24, y \in \mathrm{R}\}$

I set the mode of my calculator to radians.



I used the maximum and minimum features of my calculator.

The longest day of the year is 20.83 h.
The shortest day of the year is 4.34 h.

I knew that the maximum and minimum values represent the longest and shortest days of the year.

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b) I knew that 11:01 p.m. is 23:01 on the 24 h clock.

I used the 24 h clock to make my calculations easier. I wrote 0.01 min in hours.

$$0.01 \cancel{\text{min}} \cdot \frac{1 \text{ h}}{60 \cancel{\text{min}}} = 0.016...$$

$23.016... - 20.83$ or $2.186...$

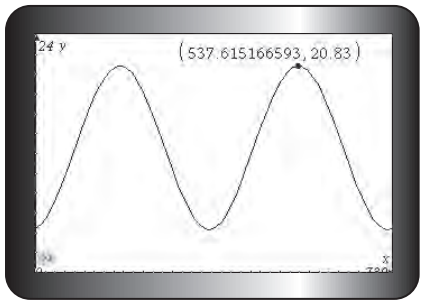
I subtracted the two times.

$$0.186... \cancel{\text{h}} \cdot \frac{60 \text{ min}}{1 \cancel{\text{h}}} = 11.2 \text{ min}$$

I wrote 0.186... in minutes.

The Sun rose at about 2:11, which is 2:11 a.m.

c) The first maximum occurs at 172.313... days.



I determined the *x*-coordinates of two consecutive maximum points on my graph. I subtracted to determine the period.

The second maximum occurs at 537.615... days.

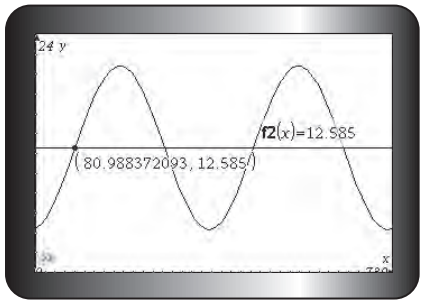
Period = second maximum − first maximum

Period = 537.615... − 172.313...

Period = 365.301...

Earth rotates about the Sun about every 365.25 days, so the period and the length of one rotation are about the same.

d) I determined the value of y when $x = 80.988$.



I noticed that 12.585 is the value of *d* in the given equation.

Based on the equation of the function, $y = d$ or $y = 12.585$ is the equation of the midline.

The point (80.988, 12.585) represents the first day of spring, when the daylight hours are getting longer. This seems reasonable, because the 81st day of the year is in March.

Your Turn

Suppose that the Sun rises at 9:22 a.m. in Iqaluit on the shortest day of the year. At what time does the Sun set?

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In Summary

Key Idea

- Any sinusoidal function can be expressed as either a cosine function or a sine function.

Need to Know

- A sinusoidal function of the form

$$y = a \sin b(x - c) + d \text{ or}$$
$$y = a \cos b(x - c) + d$$

has the following characteristics:

- The value of a is the **amplitude:**

$$a = \frac{\text{maximum value} - \text{minimum value}}{2}$$

- The value of b is the number of cycles in 360° or 2π. The **period** is $\frac{360°}{b}$ or $\frac{2\pi}{b}$.
- The value of c indicates the horizontal translation that has been applied to the graph of $y = \sin x$ or $y = \cos x$.
- The **equation of the midline** is

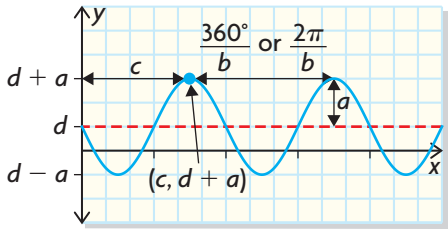
$$y = d$$

where

$$d = \frac{\text{maximum value} + \text{minimum value}}{2}$$

- The **maximum value** is $d + a$, and the **minimum value** is $d - a$.

- The graph below represents a sine curve. In the graph of a cosine curve, c is the distance from the vertical axis to the first maximum point.



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CHECK Your Understanding

1. Arrange these functions in order, from the least amplitude to the greatest amplitude. Provide your reasoning.

a) $y = 3 \sin 4x$ **b)** $y = 4 \sin 3x$ **c)** $y = 2 \sin x + 3$

2. Arrange these functions in order, from smallest range to greatest range. Provide your reasoning.

a) $y = 8 \sin 6x$ **b)** $y = 9 \sin 3x - 1$ **c)** $y = 6 \sin x + 2$

3. Arrange these functions in order, from the least period to the greatest period. Provide your reasoning.

a) $y = 5 \sin 4x$ **b)** $y = 3 \sin 2(x - 4)$ **c)** $y = 2 \cos 0.25x$

4. Describe the horizontal translation of each function.

a) $y = 0.5 \cos 2(x + 45°) + 3$

b) $y = 2 \sin 0.5(x - 180°) + 3$

c) $y = 2 \cos 2(x - 45°) + 3$

PRACTISING

5. Determine the amplitude and the range of each function.

a) $y = 7 \sin 2(x - 5)$ **b)** $y = 13 \cos 0.5(x + 26)$

6. Determine the equation of the midline, the amplitude, and the range of each function.

a) $y = 8 \sin 4(x - 22.5°) + 5$ **b)** $y = 6 \cos 3(x + 15°) - 7$

7. Determine the equation of the midline and the maximum and minimum values of each function.

a) $y = 5 \sin 2(x - 2.5) + 2$ **b)** $y = 3 \cos (x + 1) - 3$

8. Determine the distance in degrees or radians by which $y = \sin x$ or $y = \cos x$ would be horizontally translated to create the graph of each function.

a) $y = 2.5 \sin 3(x - 30°) + 5$ **c)** $y = 2 \sin 8(x - 4.5) - 3$

b) $y = 11 \cos (x + 100°) - 2$ **d)** $y = 4 \cos (x + 3) - 1$

9. Determine the period of each function and the distance by which its graph has been horizontally translated.

a) $y = 3 \sin 4(x - 45°) - 1$ **b)** $y = 2 \sin 0.5(x - 1) + 4$

10. **a)** Determine the equation of a function whose amplitude and period are both double the amplitude and period of the following function:

$$y = 3 \sin 4x$$

b) Verify your solution using graphing technology. Draw the graphs of both functions on the same axes.

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11. **a)** Determine the equation of the function whose amplitude and period are both $\frac{1}{4}$ the amplitude and period of this function:

$$y = 2 \cos x$$

b) Verify your solution using graphing technology. Draw the graphs of both functions on the same axes.

12. Ashley boards the Ferris wheel at the Pacific National Exhibition. When the ride begins, her position can be modelled by the function

$$y = 43 \sin 3.5(x - 0.9) + 47,$$

where y represents the height in feet and x represents the time in minutes.

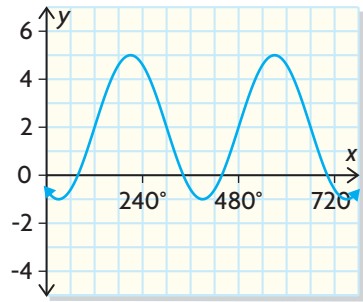
a) Determine the diameter of the Ferris wheel.

b) How long does it take for the Ferris wheel to complete one revolution?

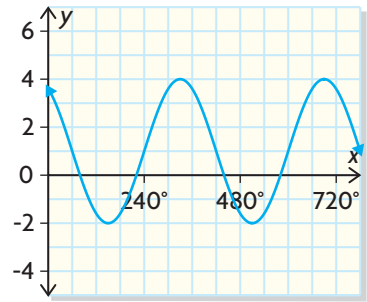
c) How high above the ground is Ashley when the ride begins?

13. Match each graph with the corresponding equation below.

a) **Graph A**



b) **Graph B**



i) $y = 3 \sin (x + 120°) + 1$

ii) $y = 3 \cos (x - 90°) + 1$

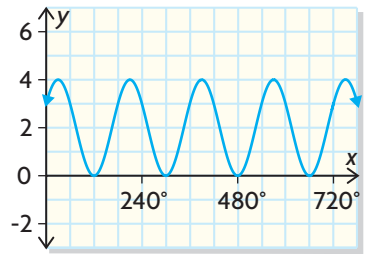
iii) $y = 3 \sin (x - 120°) + 1$

iv) $y = 4 \sin (x - 120°)$

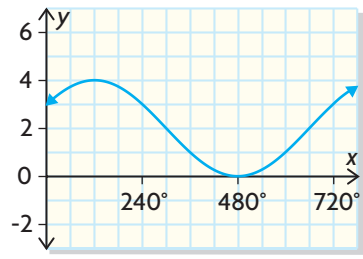
v) $y = 2 \sin (x - 60°) + 2$

14. Match each graph with the corresponding equation below.

a) **Graph A**



b) **Graph B**



i) $y = 2 \cos 2(x + 30°) + 2$

ii) $y = 2 \cos 0.5(x - 120°) + 2$

iii) $y = 3 \sin (x - 30°) + 1$

iv) $y = 2 \sin (x - 30°) + 2$

v) $y = 2 \cos 2(x - 30°) + 2$

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15. Consider the following function:

$$y = 3 \cos 5(x + 30°) - 1$$

a) Describe the graph of the function, including the amplitude, the equation of its midline, the range, the period, and the distance of horizontal translation from $y = \cos x$.

b) Confirm your description using technology.

Sierra Noble, of Winnipeg, is a Métis singer, songwriter, and fiddle player. She is one of many young Aboriginal musicians.

16. Fiddle music is traditional in Métis culture. The fiddle plays the melody in the music and tells a story. Many Métis legends are recorded in fiddle music. A fiddle has four strings, tuned to the notes G, D, A, and E. The equations that represent the sound waves of these notes are given below, where x represents time, in seconds.

G: $y = \sin 1231.5x$ A: $y = \sin 2764.6x$
D: $y = \sin 1845.4x$ E: $y = \sin 4142.5x$

a) Determine the period and the frequency (complete cycles per second) of each string. State the frequency to one decimal place.

b) Comment on the relative frequencies of the four strings.

c) If two consecutive strings are played simultaneously, how many periods will have elapsed for the sound wave from each string when the sound waves coincide at the midline and are increasing? Explain.

d) If the first three strings are played simultaneously, how many periods will have elapsed for the sound wave from each string when the sound waves coincide at the midline and are increasing?

17. a) Describe each function by stating the amplitude, the equation of the midline, the range, the period, and the horizontal translation of $y = \sin x$. Verify your descriptions using graphing technology.

i) $y = 3 \sin 2(x - 60°) + 4$

ii) $y = 3 \sin 2(x - 240°) + 4$

iii) $y = 3 \sin 2(x + 120°) + 4$

b) Compare the three graphs. What do you notice? Explain.

c) Graph the function

$$y = 3 \cos 2(x - 105°) + 4$$

Compare this graph with your graphs for part a). What do you notice? Explain.

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18. An apple is attached to a spring. The height of the apple as it oscillates up and down can be modelled by the equation

$$h(t) = 4 \sin(8\pi t) + 6.5$$

where $h(t)$ represents the height of the apple in centimetres and t represents the time in seconds.

a) What are the highest and lowest points that the apple reaches?

b) What is the period of the function? What does the period tell you about the apple in this context?

19. The height of a chair on a Ferris wheel is described by the function

$$h(t) = 15 \cos\left(t - \frac{1}{2}\right) + 18$$

where $h(t)$ represents the height of the chair in metres and t represents the time in minutes.

a) What are the maximum and minimum heights you can reach if you are riding the Ferris wheel?

b) What is the period of the function? What does the period tell you about the Ferris wheel in this context?

20. A person's blood pressure, $P(t)$, in millimetres of mercury (mm Hg), can be modelled by the function

$$P(t) = -20 \cos(8.4t) + 100$$

where t is the time in seconds.

a) What is the period of the function?

b) What does the value of the period mean in this situation?

Closing

21. Explain how to describe the graph of a sinusoidal function, given its equation. Give an example.

Extending

22. **a)** Graph the functions

$$y = 2 \sin x \text{ and}$$
$$y = -2 \sin x$$

on the same axes. How are your graphs the same? How are they different?

b) What horizontal translation of $y = 2 \sin x$ has the same graph as $y = -2 \sin x$?

c) Do you think what you observed in part b) would always be the same when the values of a are opposites? Explain.

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Math in Action

Biorhythms

How do you feel today? Think about how you feel physically, emotionally, intellectually, and intuitively. Do you feel better than usual? Worse than usual? Did you feel the same yesterday? How will you feel tomorrow?

Biorhythm charts are based on the assumption that four cycles contribute to our sense of well-being: physical, emotional, intellectual, and intuitive. According to the people who make these charts, all the cycles start from zero at birth (day 0) and then rise in an active phase before descending through zero into a passive phase.

Cycle	Period (days)	Affected Attributes
physical	23	strength, stamina
emotional	28	moods, mental stability, sensitivity
intellectual	33	reasoning, memory
intuitive	38	instincts, perception

- Use sinusoidal functions to model the four biorhythm cycles for the first 100 days after a person is born.
- How many days after a person is born will all four cycles again be at zero at the same time? How old will the person be?
- Is it possible for all four cycles to be at a maximum or a minimum at the same time? Explain.
- Taking into account leap years, which cycle(s) should be near maximum for you today? Which cycle(s) should be near minimum? Do the predictions of the biorhythm charts match how you feel?

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8.5 Modelling Data with Sinusoidal Functions

GOAL

Determine the sinusoidal function that best models a set of data, and use your model to solve a problem.

YOU WILL NEED

- graphing technology

EXPLORE...

- Think of naturally occurring phenomena that you can model with a sinusoidal function. How many phenomena can the class come up with?

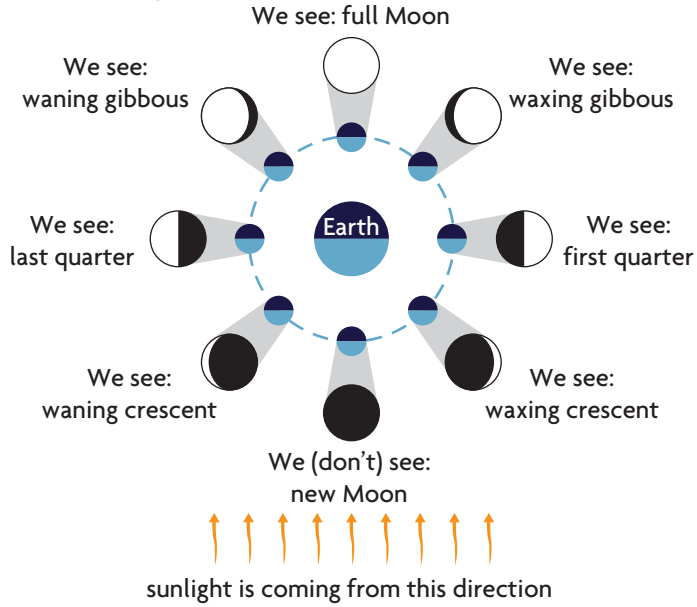
INVESTIGATE the Math

Kelly lives in Winnipeg, Manitoba. She walks her dog at the same time each evening. She noticed that a different percent of the Moon is illuminated each evening, so she decided to look for patterns. She recorded the following data for three months, beginning on April 1.

Percent of the Moon Visible Each Evening

Day	1	3	5	7	9	11	13	15	17	19	21	23
Percent (%)	2	0	6	20	41	64	85	96	100	95	82	63
Day	25	27	29	31	33	35	37	39	41	43	45	47
Percent (%)	43	25	10	1	0	2	12	30	52	75	93	98
Day	49	51	53	55	57	59	61	63	65	67	69	71
Percent (%)	99	89	72	53	33	17	5	0	4	17	38	61
Day	73	75	77	79	81	83	85	87	89	91		
Percent (%)	83	97	100	97	86	69	50	31	14	3		

? Kelly is planning a camping trip in July, and she wants her trip to occur during a full Moon. When can she expect to see a full Moon in July?


We see: full Moon
We see: waning gibbous
We see: waxing gibbous
We see: last quarter
Earth
We see: first quarter
We see: waning crescent
We see: waxing crescent
We (don't) see: new Moon
sunlight is coming from this direction

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A. Create a scatter plot on graph paper. Indicate the day along the horizontal axis and the percent of the Moon illuminated along the vertical axis. Draw a curve of best fit.

B. How would you describe your graph? Explain.

C. The synodic period is the length of time between full Moons. Estimate the synodic period using your graph.

D. Enter the data into your graphing calculator. Create a scatter plot.

E. Determine the equation of the sinusoidal regression function that models the data.

F. Comment on the accuracy of your regression equation.

G. Determine the synodic period using your regression equation.

H. On what date in July will the full Moon occur? Explain.

Reflecting

I. The synodic period of the Moon is actually 29.53 days. Compare your result with this value.

J. Would you rely on a prediction made from Kelly's data? Explain.

K. To save time, Betty entered only every fourth data point in her graphing calculator. Would she get the same sinusoidal regression equation that you did? Would Betty's prediction be as reliable? Explain.

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APPLY the Math

EXAMPLE 1 Solving an interpolation problem using a sinusoidal model

Celeste lives in Red Deer, Alberta. The predicted hours of daylight for two consecutive years are shown in the tables below. In the second year, the spring equinox will occur on March 20 and the fall equinox will occur on September 23. Compare the hours of daylight on these two days.

Hours of Daylight in Red Deer This Year		
Date	**Day Number**	**Length of Day (h)**
Jan. 1	1	7.812
Feb. 1	32	9.113
Mar. 1	60	10.896
Apr. 1	91	12.998
May 1	121	14.944
Jun. 1	152	16.455
Jul. 1	182	16.690
Aug. 1	213	15.494
Sep. 1	244	13.595
Oct. 1	274	11.597
Nov. 1	305	9.580
Dec. 1	335	8.064

Predicted Hours of Daylight in Red Deer Next Year		
Date	**Day Number**	**Length of Day (h)**
Jan. 1	366	7.808
Feb. 1	397	9.100
Mar. 1	425	10.880
Apr. 1	456	12.982
May 1	486	14.929
Jun. 1	517	16.447
Jul. 1	547	16.694
Aug. 1	578	15.507
Sep. 1	609	13.611
Oct. 1	639	11.613
Nov. 1	670	9.595
Dec. 1	700	8.073
Jan. 1	731	7.803

Celeste's Solution

I entered the data into a spreadsheet.

	A	B
1	Day Number	Length of Day (h)
2	1	7.812
3	32	9.113
4	60	10.896
25	700	8.073
26	731	7.803

I entered the day numbers in column A, so the independent variable would precede the dependent variable.

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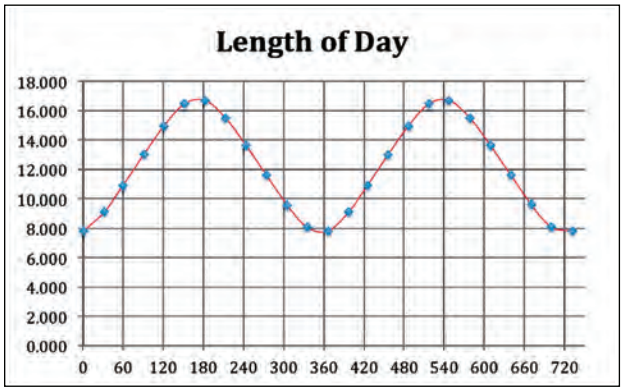
I created a graph I could use to compare day number to hours of daylight.



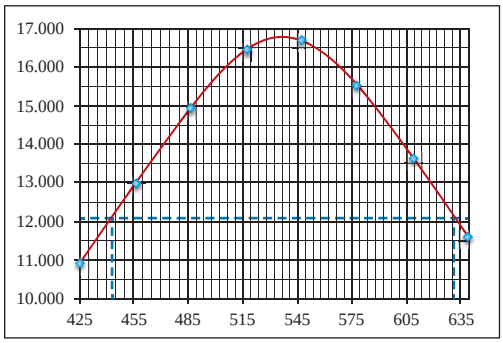
I saw that the graph formed a sine wave. This makes sense because the hours of daylight are determined by the tilt of the Earth as it orbits the Sun in a repeating cycle.

I determined the period.

The data is sinusoidal.
One maximum point is at about $x = 180$ and the next maximum point is at about $x = 545$.
Therefore, the period is about $545 - 180$, or 365 days. This makes sense, because a year is 365 days long.

I determined the day numbers for the spring and fall equinoxes.

March 1 of next year is day 425, so March 20 will be 19 days later, or day 444.
September 1 of next year is day 609, so September 23 will be 22 days later, or day 631.



I zoomed in on the area of the graph that I wanted to study.

I determined the points on the graph for both dates. I looked at the *y*-coordinate of each point.

By interpolation, I can see that days 444 and 631 both have about 12.1 h of daylight.

The spring and fall equinoxes both have about the same amount of daylight, about 12.1 h.

These values are approximate, because I interpolated the data on a graph.

Your Turn

Suppose that the summer solstice will be on June 21 and the winter solstice will be on December 21 this year. Determine the hours of daylight in Red Deer on each day. Round to two decimal places.

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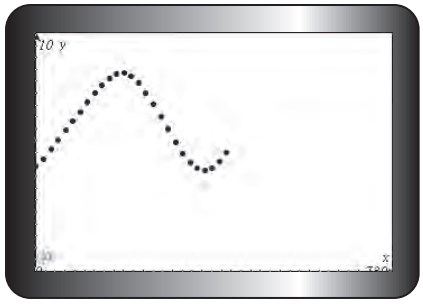
EXAMPLE 2 Solving an extrapolation problem using a sinusoidal model

Tara's high school class is planning a trip to British Columbia to study eco-tourism next year. The students want to finish the tour, on May 21, by watching the Sun set over the Pacific Ocean. Tara found the following data for the current year and the following January. All the times are in standard time, as decimal values. At what time should the class be at the vantage point on the cliff?

Time of Sunset on West Coast of British Columbia Next Year								
Date	**Day**	**Time of Sunset**	**Date**	**Day**	**Time of Sunset**	**Date**	**Day**	**Time of Sunset**
Jan. 1	1	4.42	May 16	136	7.83	Sep. 28	271	5.97
Jan. 16	16	4.72	May 31	151	8.13	Oct. 13	286	5.43
Jan. 31	31	5.12	Jun. 15	166	8.32	Oct. 28	301	4.93
Feb. 15	46	5.53	Jun. 30	181	8.35	Nov. 12	316	4.58
Mar. 2	61	5.95	Jul. 15	196	8.22	Nov. 27	331	4.32
Mar. 17	76	6.33	Jul. 30	211	7.92	Dec. 12	346	4.23
Apr. 1	91	6.72	Aug. 14	226	7.52	Dec. 27	361	4.33
Apr. 16	106	7.1	Aug. 29	241	7.03	Jan. 11	376	4.6
May 1	121	7.48	Sep. 13	256	6.5	Jan. 26	391	4.98

Tara's Solution

I entered the data into my graphing calculator and created a scatter plot.



I think there might be minimum points when $x = 0$ and $x = 365$.

I examined the points. It made sense that the period is 365 days or 1 year.

The period is about 365 days or about 1 year.

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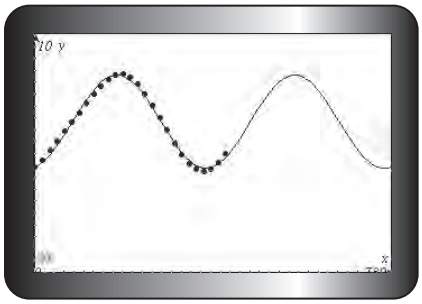
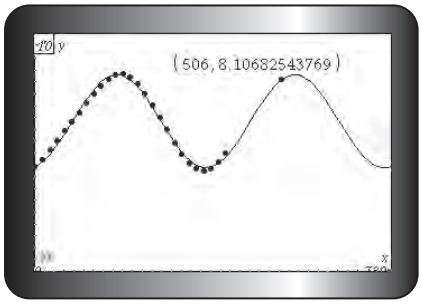


I determined the equation of the sinusoidal regression function and graphed the function.

$y = 1.966... \sin(0.0172... x - 1.334...) + 6.342...$

May 21 is day 141 of this year. Next year, May 21 will be 365 + 141, or day 506.

If next year was going to be a leap year, I would have added 366 instead of 365 to the day number.



I determined the time when the Sun will set on May 21 of next year using my calculator.

$0.107... \text{ 9 h} \cdot \frac{60 \text{ min}}{1 \text{ h}} = 6.4 \text{ min}$

8.107... h is equivalent to 8:06 p.m. standard time, to the nearest minute.

I converted the time from a decimal in hours to hours and minutes.

I added an hour to account for daylight-saving time.

The Sun will set at 8:06 p.m. standard time. Since British Columbia is on daylight-saving time in May, the Sun will set at 9:06 p.m.

We should arrive at the vantage point on the cliff at least 10 min before 9:06 to be sure that we do not miss the sunset.

This value is an estimate, since I extrapolated using a graph.

Your Turn

Suppose that Tara's class decides to finish its tour on June 25 instead of May 21. At what time should they arrive at the vantage point on the cliff?

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EXAMPLE 3 Solving a problem using a sinusoidal model

In 2011, the Singapore Flyer was the largest Ferris wheel in the world. The table below gives the height of a rider from the ground at different times.

Time (min)	Height (ft)
0	49
9.25	295
18.50	541
27.75	295
37.00	49
46.25	295
55.50	541
64.75	295
74.00	49

The Singapore Flyer, built on land reclaimed from the sea, first turned in 2008.

Jordy got on the Singapore Flyer at noon and rode it for four consecutive rotations. His friend Yale was in a building directly across from the Singapore Flyer, at a height of 400 ft. When was Jordy level with Yale?

Yale's Solution

I entered the given data into my calculator and created a scatter plot.

The minimum heights are at 0, 37, and 74 min.
The period appears to be about 37 min.

I examined the points.

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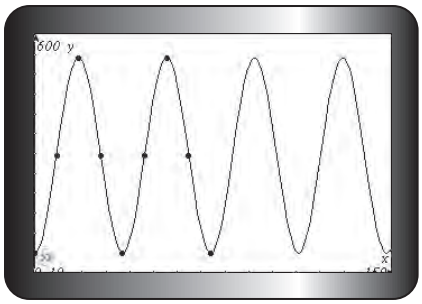
I determined the equation of the sinusoidal regression function:

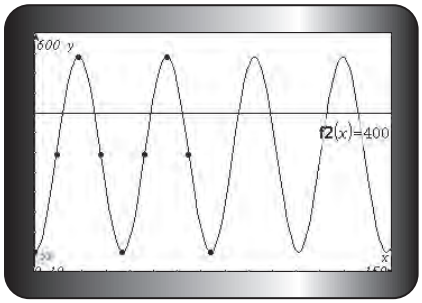
$y = 246 \sin (0.169... x - 1.570...) + 295$

I graphed the function.



Jordy rode the Ferris wheel for four rotations, so I adjusted the window to show just over four periods.

I added a graph of the horizontal line $y = 400$.



This line crosses the graph eight times.

I determined all the intersection points of the line and the sine wave.

The two graphs intersect at 11.85 min, 25.15 min, 48.85 min, 62.15 min, 85.85 min, 99.15 min, 122.85 min, and 136.15 min.

I verified the intersection points.

I verified my results. If they are correct, then adding one period to any time will result in another time when the Ferris wheel is in the same position.

11.85 min + 37 min or 48.85 min
48.85 min + 37 min or 85.85 min
85.85 min + 37 min or 122.85 min

25.15 min + 37 min or 62.15 min
62.15 min + 37 min or 99.15 min
99.15 min + 37 min or 136.15 min

The intersection points are correct.

Jordy was level with me at the following times after the Ferris wheel started:

11.85 min	85.85 min
25.15 min	99.15 min
48.85 min	122.85 min
62.15 min	136.15 min

Your Turn

At what time was Jordy at a height of 500 ft for the fifth time?

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In Summary

Key Idea

- If the data points on a scatter plot seem to follow a regular periodic pattern of increasing and decreasing curves, then there may be a sinusoidal relationship between the independent and dependent variables.

Need to Know

- If the points on a scatter plot show a sinusoidal trend, then graphing technology can be used to determine the equation of the sinusoidal regression function that models the data. Use radian mode when performing sinusoidal regression.
- Interpolated or extrapolated values can be predicted by reading values from a graph or by using the equation of the sinusoidal regression function.
- When the data in a set repeats in a regular, periodic pattern, interpolation or extrapolation can be used to make predictions.

CHECK Your Understanding

1. A cuckoo clock is hanging on a wall. The table below gives the height of the pendulum above the ground as the clock ticks.

Time (s)	0	0.5	1.0	1.5	2.0	2.5	3.0
Height (in.)	65.0	66.5	68.0	66.5	65.0	66.5	68.0

a) Plot the data.

b) Determine the equation of a sinusoidal regression function that models the movement of the pendulum.

2. A water wheel, with a radius of 5 ft, is 1 ft above the ground. When water falls onto the wheel, the wheel completes one revolution every 2 s.

a) Complete the table for a point on the water wheel as it rotates.

Revolution	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2
Time (s)	0		1		2				4
Height (ft)	11		1		11				

b) Determine the equation of a sinusoidal regression function that models the data.

c) How high is the point at 1.15 s? State your answer in feet and inches, to the nearest inch.

d) How high is the point at 7.2 s? State your answer in feet and inches, to the nearest inch.

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PRACTISING

3. While on her nightly walks with her dog, Kelly noticed that the Moon appeared to be at different angles on different days. She researched the maximum altitude of the Moon, in degrees, and obtained this data, starting on April 1.

Day	1	3	5	7	9	11	13	15	17	19
Angle (°)	41.2	50.6	58.4	63	63.1	57.9	48.2	36.1	30.1	20.5
Day	21	23	25	27	29	31	33	35	37	39
Angle (°)	16.7	19.1	25.9	34.9	44.5	53.4	60.3	63.5	61.7	54.9
Day	41	43	45	47	49	51	53	55	57	59
Angle (°)	44.5	32.7	24.5	19.0	16.9	20.8	28.6	38.0	47.5	56.0

a) Graph Kelly's data, and estimate the period.
b) Explain why a sinusoidal function can be used to model the data.
c) Determine the equation of a sinusoidal regression function that models the data.
d) What is the maximum angle of the Moon?
e) Estimate the altitude of the Moon, in degrees, on day 100. Round to the nearest tenth of a degree.
f) What is the first day in May that the altitude of the Moon will drop below 30°?

In 2011, the Bay of Fundy was under consideration as one of the Seven Natural Wonders of North America.

4. The Bay of Fundy, in the Maritimes, has the highest tides in the world. The height of the water, in metres above the seabed, is shown for one point over 36 h.

Hour	Height (m)	Hour	Height (m)	Hour	Height (m)	Hour	Height (m)
1	3.5	10	7.0	19	4.6	28	1.6
2	2.4	11	6.4	20	5.9	29	1.9
3	1.8	12	5.3	21	6.9	30	2.7
4	1.9	13	4.0	22	7.2	31	4.0
5	2.7	14	2.7	23	6.8	32	5.4
6	3.8	15	1.8	24	5.9	33	6.5
7	5.2	16	1.7	25	4.6	34	7.1
8	6.3	17	2.2	26	3.2	35	7.0
9	7.0	18	3.3	27	2.1	36	6.3

a) Explain why a sinusoidal function can be used to model the data.
b) Graph the data. Determine the equation of a sinusoidal regression function that models the height of the water. Does the regression equation match the data closely?

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c) How high is the water at high tide, to the nearest tenth of a metre? How high is the water at low tide?

d) How long, to the nearest minute, does it take for the tide to cycle from high tide to low tide and back again?

e) Simon plans to go fishing at hour 50. How high, to the nearest tenth of a metre, will the tide be when he begins fishing?

5. A boat needs to be in water that is at least 4 m deep to move safely. For how many consecutive hours is it safe for this boat to sail in the Bay of Fundy? (Use the data in question 4.)

6. The data for monthly temperatures in Winnipeg is given in the table below.

Month	Jan.	Feb.	Mar.	Apr.	May	Jun.	Jul.	Aug.	Sep.	Oct.	Nov.	Dec.
Average High (°C)	−12.2	−5.2	0.3	10.7	17.6	21.8	25.4	25.3	19.1	10.8	−1.3	−7.5
Average Low (°C)	−21.3	−15.7	−10	−2.6	3.4	9.9	12.3	11.6	5.6	−0.4	−10.3	−17.2
Average (°C)	−16.9	−10.4	−4.9	4.1	10.5	15.9	18.9	18.5	12.4	5.2	−5.9	−12.4
Record Daily High (°C)	4.9	8.5	16.6	28.1	31.8	31.4	32.3	33	31.9	26.3	17.2	8.9

a) Determine the equation of a sinusoidal regression function that models the average high temperatures.

b) Determine the equation of a sinusoidal regression function that models the average low temperatures.

c) Determine the equation of a sinusoidal regression function that models the average temperatures.

d) Compare the three equations you determined in parts a) to c). How are they similar? How are they different?

e) Do you think that the record daily high temperatures form a sinusoidal pattern? Explain.

f) Estimate the average high temperature on October 1, to the nearest tenth of a degree Celsius. Assume that the average monthly temperature is the temperature on the 15th of the month.

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7. Sally will be travelling to Yellowknife in June of next year. She collected the following data for average temperatures.

Average Temperatures Last Year			Average Temperatures This Year		
Month	Month Number	Average Temperature (°C)	Month	Month Number	Average Temperature (°C)
Jan.	1	–25.6	Jan.	13	–24.2
Feb.	2	–23.5	Feb.	14	–16.3
Mar.	3	–20.3	Mar.	15	–9.9
Apr.	4	–5.0	Apr.	16	0.8
May	5	1.4	May	17	5.6
Jun.	6	12.2	Jun.	18	14.8
Jul.	7	15.8	Jul.	19	17.9
Aug.	8	14.8	Aug.	20	15.5
Sep.	9	10.4	Sep.	21	7.8
Oct.	10	–1.6	Oct.	22	0.4
Nov.	11	–10.9	Nov.	23	–12.0
Dec.	12	–22.0	Dec.	24	–21.9

a) Determine the equation of a sinusoidal regression function that models the average temperatures.

b) Determine the maximum and minimum values of the sinusoidal regression function. What do these values show about the accuracy of the regression function?

c) Based on your sinusoidal regression function, what will be the average temperature in June of next year?

Yellowknife, situated on Great Slave Lake within the traditional territories of the Yellowknives Dene First Nation, was founded by gold prospectors in the 1930s. The town is also known as Somba k'e, "where the money is" in Dogrib.

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8. The following table gives the hours of daylight in Regina, which is at a latitude of 50°. The hours of daylight are calculated as the time between sunrise and sunset each day.

Day of Year	15	46	74	105	135	165	196	227	258	288	319	349
Hours of Daylight	8.5	10.1	11.8	13.7	16.4	17.1	15.6	14.6	12.7	10.8	9.1	8.1

a) Determine the equation of a sinusoidal regression function that models the data.

b) Determine the range between the maximum and minimum hours of daylight in Regina. Round to the nearest minute.

c) Which day has the most hours of daylight?

d) Determine the hours of daylight in Regina on day 30. Round to the nearest minute.

e) Determine which days of the year have about 15.0 h of daylight.

The Saskatchewan Legislative Building, completed in 1912, is set in a park on Lake Wascana in Regina.

9. This table gives the average monthly temperature and precipitation in Edmonton.

Month	Jan.	Feb.	Mar.	Apr.	May	Jun.	Jul.	Aug.	Sep.	Oct.	Nov.	Dec.
Average Monthly Temperature (°C)	–13.9	–5.4	–3.6	5.3	10.6	14.7	17.4	17.3	11.6	5.3	–4	–8.7
Average Monthly Precipitation (mm)	22	9	26	36	57	88	66	53	45	17	23	13

a) Predict what the average temperature will be next October, to the nearest tenth of a degree Celsius. Explain what you did.

b) Predict what the average precipitation will be next October, to the nearest millimetre. Explain what you did.

c) Which of your two predictions do you think is more reliable? Explain your answer.

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10. Catherine swings a pendulum back and forth. Its minimum height from the floor is 40 cm, and its maximum height is 70 cm. The pendulum falls from the highest point to the lowest point in 0.5 s.

a) Create a table of values for height versus time, for $t = 0$ to $t = 3$ s.

b) Determine the equation of the sinusoidal regression function that models the height of the pendulum over time.

c) How high will the pendulum be after 7.25 s?

11. During a physics experiment, Carol held down a weight attached to a spring and then released the weight. The height of the weight above the table is shown.

Time (s)	0	0.5	1	1.5	2	2.5	3	3.5	4
Height (in.)	5	11	17	11	5	11	17	11	5

a) Determine the equation of a sinusoidal function that models the data.

b) How high above the table will the spring be after 0.75 s? How high will the spring be after 5.3 s? Round your answers to the nearest tenth.

Closing

12. This table gives the average monthly temperature for Prince George, British Columbia.

Month	Jan.	Feb.	Mar.	Apr.	May	Jun.	Jul.	Aug.	Sep.	Oct.	Nov.	Dec.
Average Monthly Temperature (°C)	–8	–3.9	0.9	6.1	11	14.4	16.5	16	11.3	5.7	–1.6	–6.5

Jerry says, "To predict the average monthly temperature for Prince George next November, I just have to look at the table. The temperature will be −1.6 °C. I don't need to bother doing a regression." Do you agree with Jerry? Explain.

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Extending

13. The average monthly temperatures, in degrees Celsius, for two communities are given.

Month	Jan.	Feb.	Mar.	Apr.	May	Jun.	Jul.	Aug.	Sep.	Oct.	Nov.	Dec.
Hay River, NWT (°C)	−24.5	−21.2	−15.8	−3.5	5.9	12.3	15.8	14.4	8.2	0.9	−12.4	−20.9
Mar del Plata, Argentina (°C)	20.3	19.9	18	14.6	11.3	8.5	8.1	8.9	10.5	13.1	15.9	18.5

a) Determine the equation of a sinusoidal regression function that models the temperatures in each city. Does each function fit the data closely?
b) Compare the graphs of the two functions.
c) How does the location of each community affect its average monthly temperatures?
d) On about what dates would you expect the two communities to have the same temperature? Explain.
e) The average monthly temperatures, in degrees Celsius, for City X are given below. Is this city north or south of the equator? Provide your reasoning.

Month	Jan.	Feb.	Mar.	Apr.	May	Jun.	Jul.	Aug.	Sep.	Oct.	Nov.	Dec.
City X (°C)	22	23	22.5	21	18.5	17	15.5	15	15	16.5	18	20

14. The word "month" comes from the word "Moon." Therefore, you might think that the Moon goes through 12 cycles in a year, since there are 12 months in a year. However, the lunar year is actually made up of about 13 cycles as its sidereal period is 27.32 days, taking into consideration that the Moon is both rotating about its axis and revolving about the Earth simultaneously.
a) Why do you think that the sidereal period is shorter than the synodic period (the time from full moon to full moon)?
b) How does the graph of the sidereal period compare with the graph of the synodic period?

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Applying Problem-Solving Strategies

YOU WILL NEED

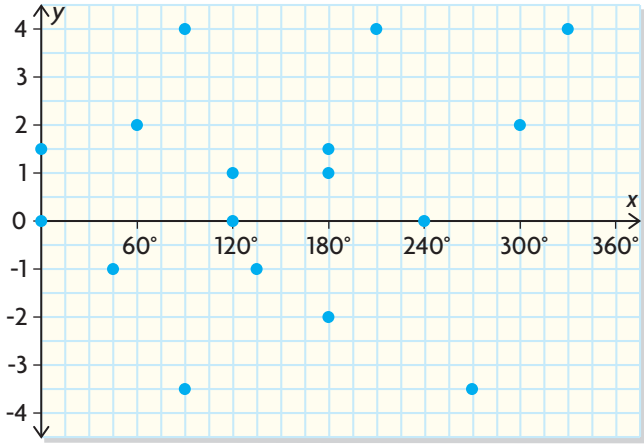
- blackline master
- coloured pencils or markers

Hidden Waves

Dominika graphed three sinusoidal functions for her math class. Her younger brother erased the graphs by mistake, leaving behind only the points shown. All of these points are either maximum or minimum values, or on the midline of a sinusoidal graph. Dominika needs to recreate each graph.

The Puzzle

What are the graphs of the three sine functions?



The Strategy

A. What points would you expect to lie on the same horizontal line for each graph? How could you use this information to determine which points belong to each graph?

B. How could determining the points that have x-values halfway between the x-values of two other points help you determine which points belong to each graph?

C. Continue looking for patterns until you have recreated all three graphs.

Variation

D. Draw two or three sinusoidal functions, and select six points from each function. Plot these points on a separate grid to make a puzzle. Exchange puzzles with a classmate, and see if you can solve each other's puzzles.

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8 Chapter Self-Test

1. Estimate, to the nearest tenth, each measure in radians.
a) 186° **b)** 500° **c)** 20° **d)** 345°

2. Estimate, to the nearest degree, each measure in degrees.
a) 5.6 **b)** 4.71 **c)** 6 **d)** 9.5

3. Determine the x-intercepts of the graph of the function
$$y = 2 \sin 3x$$
over each interval:
a) from 0° to 720° **b)** from 0 to 12.6

4. Determine the range, amplitude, period, horizontal translation of $y = \sin x$ and equation of the midline of each function.
a) $y = 4 \sin 3(x - 20°) + 3$ **b)** $y = 2 \sin 2(x - 60°) - 4$

5. The average high monthly temperatures for Dauphin, Manitoba, are given in the table below. Determine the equation of a sinusoidal regression function that models these temperatures.

Month	Jan.	Feb.	Mar.	Apr.	May	Jun.	Jul.	Aug.	Sep.	Oct.	Nov.	Dec.
Average High (°C)	−11.9	−7.8	−0.9	9.5	18.2	22.4	24.9	24.2	17.5	10.4	−0.9	−9.3

6. A coach measures the velocity of air as a gymnast inhales and exhales after working out.
a) Plot the data in the table, and determine the equation of a sinusoidal function that models the data.
b) State the period of the function.
c) Explain why some velocities are positive and other velocities are negative.
d) Predict when the velocity of air will be 0 L/s from 9 s to 19 s.

Time (s)	Velocity of Air (L/s)	Time (s)	Velocity of Air (L/s)	Time (s)	Velocity of Air (L/s)
1.0	1.75	4.0	0	7.0	–1.75
1.5	1.24	4.5	1.24	7.5	–1.24
2.0	0	5.0	1.75	8.0	0
2.5	–1.24	5.5	1.24	8.5	1.24
3.0	–1.75	6.0	0	9.0	1.75
3.5	–1.24	6.5	–1.24		

***WHAT** DO You Think Now?* Revisit **What Do You Think?** on page 483. How have your answers and explanations changed?

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8 Chapter Review

FREQUENTLY ASKED Questions

Study Aid
- See Lesson 8.4, Examples 1, 2, and 3.
- Try Chapter Review Questions 8 and 9.

Q: How can you determine the characteristics of a sinusoidal function from its equation?

A: Sinusoidal equations can be written in the following form:

$$y = a \sin b(x - c) + d$$

Characteristics of $y = a \sin b(x - c) + d$	Example $y = 3 \sin 1.256(x - 1.5) + 1$
• The amplitude of the function is a.	Amplitude: 3
• The equation of the midline is $y = d$.	Equation of the midline: $y = 1$
• The maximum value is the sum of the value of the midline and the amplitude, or $d + a$.	Maximum value: $1 + 3$, or 4
• The minimum value is the difference between the value of the midline and the amplitude, or $d - a$.	Minimum value: $1 - 3$, or -2
• The range is $\{y \mid (d - a) \leq y \leq (d + a), y \in \mathrm{R}\}$.	Range: $\{y \mid -2 \leq y \leq 4, y \in \mathrm{R}\}$
• The horizontal translation from $y = \sin x$ is c.	Horizontal translation: 1.5
• The period is $\frac{2\pi}{b}$.	Period $= \frac{2\pi}{1.256}$ or 5.002...

Study Aid
- See Lesson 8.5, Examples 1 to 3.
- Try Chapter Review Questions 11 and 12.

Q: How can you model periodic data and use your model to solve a problem?

A: Draw a scatter plot to display the data. Determine the equation of a sinusoidal regression function that models the data, and graph the function. Check that your graph is a reasonable fit to the data.

To determine the x-values for a given y-value, graph a horizontal line at the given y-value and determine the intersection points.

To determine the y-value for a given x-value, use the table feature, or identify a point on the sinusoidal function with the given x-value, or substitute x into the regression equation and solve.

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PRACTISING

Lesson 8.1

1. Estimate, to the nearest tenth, each measure in radians.

a) 15°
b) 113°
c) 220°
d) 555°

2. Estimate to the nearest degree, each measure in degrees.

a) 9.5
b) 5.24
c) 0.5
d) 11.0

Lesson 8.2

3. **a)** For the function $y = \sin x$, determine the values of x where $\sin x < 0$ for the interval 0° to 720°.

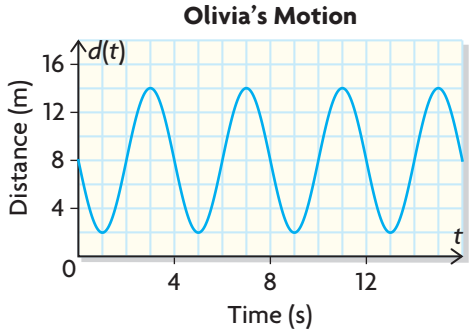
b) For the function $y = \cos x$, determine the values of x where $\cos x > 0$ for the interval 0° to 720°.

Lesson 8.3

4. Determine the range, amplitude, period, and equation of the midline of each function.

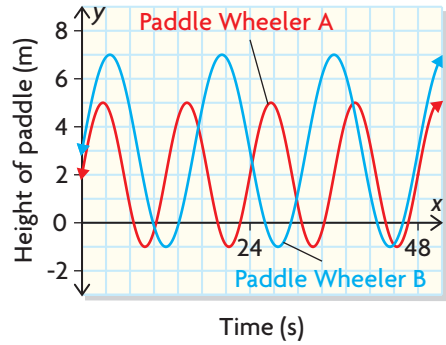
a) $y = 2 \sin (x + 60°) - 2$
b) $y = \sin 4(x - 30°) - 1.5$

5. Olivia was swinging back and forth in front of a motion detector. Her distance from the detector, in terms of time, can be modelled by the graph shown.



a) What is the equation of the midline? What does it represent in this situation?
b) What is the amplitude of the function?
c) What is the period of the function? What does it represent in this situation?
d) How close did Olivia get to the motion detector?
e) At $t = 7$ s, would it be safe to run between Olivia and the motion detector? Provide your reasoning.

6. Marianna collected data on the revolution of two paddle wheels on two different boats. She used the data to construct two graphs.



a) Determine the radius of each wheel, the height of the axle relative to the water, and the time taken to complete one revolution.
b) Which paddle wheel is travelling faster? Explain.

7. When an emergency vehicle approaches you at 30 m/s, the sound of the siren can be modelled by the equation

$$y = \sin 5510.35\, x$$

where x represents the time in seconds. As the vehicle drives away, the sound can be modelled by the equation:

$$y = \sin 4618.14\, x$$

a) Graph each equation. Does the siren have a higher frequency as it approaches you or as it drives away from you?
b) What is the frequency, in cycles per second (Hz), for each equation?

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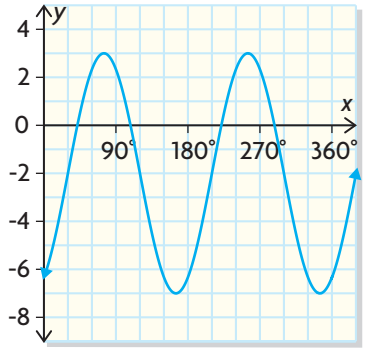
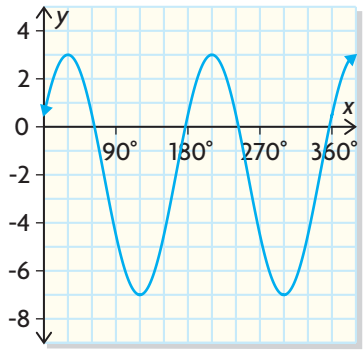
Lesson 8.4

8. Match each graph with the corresponding equation below.

a) **Graph A**



b) **Graph B**



i) $y = 3 \sin 2(x - 30°)$
ii) $y = 5 \cos 2(x - 30°) - 2$
iii) $y = 5 \cos 2(x + 75°) - 2$
iv) $y = 5 \sin 2(x - 30°) - 2$

9. Consider the function

$$y = 4 \sin 6(x - 2.5) + 3$$

a) Describe the graph of this function by stating the amplitude, equation of the midline, range, period, and horizontal translation of $y = \sin x$.

b) Verify your description using technology.

10. The signals from WiFi transmitters can be modelled using sinusoidal functions. Two such functions are

$$y = \sin 15.08\, x \text{ and}$$
$$y = \sin 31.42\, x$$

where the signal, y, is measured in gigahertz (GHz) and the time, x, is measured in nanoseconds.

What is the frequency of each of the WiFi signals above, in cycles per nanosecond?

Lesson 8.5

11. The average monthly temperatures for Humboldt, Saskatchewan, are given in the table below. Predict the average temperature in Humboldt on December 8.

Month	Average (°C)	Month	Average (°C)
Jan.	–17.9	Jul.	17.4
Feb.	–13.6	Aug.	16.7
Mar.	–6.8	Sep.	10.7
Apr.	3.2	Oct.	3.9
May	10.8	Nov.	–7
Jun.	15.3	Dec.	–15.2

12. The average monthly low temperatures in Edmonton, Alberta, and Rio de Janeiro, Brazil, are given in the table below.

Month	Edmonton (°C)	Rio de Janeiro (°C)
Jan.	–19.1	23.3
Feb.	–16.3	23.5
Mar.	–9.9	23.3
Apr.	–2.2	21.9
May	3.4	20.4
Jun.	7.7	18.7
Jul.	9.5	18.4
Aug.	8.3	18.9
Sep.	3.3	19.2
Oct.	–2.4	20.2
Nov.	–11.0	21.4
Dec.	–16.7	22.4

Predict the difference in the average low temperatures for the two cities on June 28.

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8 Chapter Task

Here Comes the Sun

Have you ever wondered about the amount of sunlight in different areas of the world and how it compares with the amount of sunlight where you live?

How does the amount of sunlight differ in different locations?

A. Choose five locations: where you live, in northern Canada, in southern Canada, near the equator, and in the southern hemisphere.

B. Choose the type of data you would like to compare: sunrise, sunset, or daylight hours. (Use standard time.)

C. Determine the latitude of each location you have chosen.

D. Research the required data for each location over a period of one year. Decide how many data points you require for each location. Adjust your data for daylight-saving time, if applicable.

E. Graph a scatter plot for each location. Determine the equation of the sinusoidal regression function that models each set of data.

F. State the equation of the midline, the range, and the amplitude of each function.

G. Compare your locations. Can you use your data to compare the climates?

H. Compare your home location with the location in the southern hemisphere. Can you use your data to describe the seasons in the southern hemisphere?

Task Checklist

- ✔ Did you include your data for each location?
- ✔ Did you clearly label each graph?
- ✔ Did you explain each comparison clearly?
- ✔ Did you use appropriate mathematical language?

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8 Project Connection

Peer Critiquing of Research Projects

Now that you have completed your research for your question/topic, prepared your report and presentation, and assessed your own work, you are ready to see and hear the research projects developed by your classmates. You will not be a passive observer, however. You will have an important role—to provide constructive feedback to your peers about their projects and their presentations.

Critiquing a project does *not* involve commenting on what might have been or should have been. It involves reacting to what you see and hear. For example, pay attention to

- strengths and weaknesses of the presentation
- problems or concerns with the presentation

Think of suggestions you could make to help your classmate improve future presentations.

While observing each presentation, consider the content, the organization, and the delivery. Take notes during the presentation, using the following rating scales as a guide. You can also use these scales to evaluate the presentation.

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Content

Shows a clear sense of audience and purpose.	5	4	3	2	1
Demonstrates a thorough understanding of the topic.	5	4	3	2	1
Clearly and concisely explains ideas.	5	4	3	2	1
Applies mathematical knowledge and skills developed in this course.	5	4	3	2	1
Justifies conclusions with sound mathematical reasoning.	5	4	3	2	1
Uses mathematical terms and symbols correctly.	5	4	3	2	1

Organization

Presentation is clearly focused.	5	4	3	2	1
Engaging introduction includes the research question, clearly stated.	5	4	3	2	1
Key ideas and information are logically presented.	5	4	3	2	1
There are effective transitions between ideas and information.	5	4	3	2	1
Conclusion follows logically from the analysis and relates to the question.	5	4	3	2	1

Delivery

Speaking voice is clear, relaxed, and audible.	5	4	3	2	1
Pacing is appropriate and effective for the allotted time.	5	4	3	2	1
Technology is used effectively.	5	4	3	2	1
Visuals and handouts are easily understood.	5	4	3	2	1
Responses to audience's questions show a thorough understanding of the topic.	5	4	3	2	1

Keep these rating scales in mind as you prepare your own presentation.

Your Turn

A. Think of an excellent presentation you have observed. List five aspects of this presentation that made it effective.

B. What are some common difficulties that students encounter when giving presentations to their peers? Suggest how each of these difficulties could be avoided or dealt with.

C. When critiquing a presentation, it is essential that you do not allow your own opinions about the topic to influence your analysis. Explain why.

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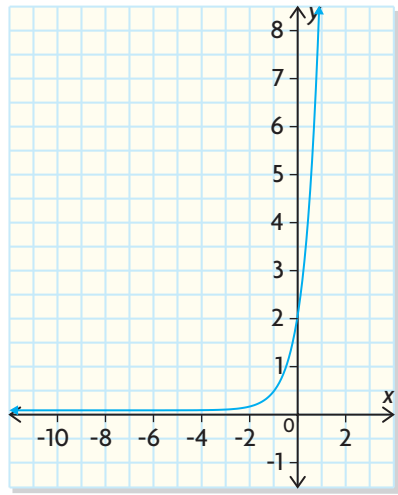
6-8 Cumulative Review

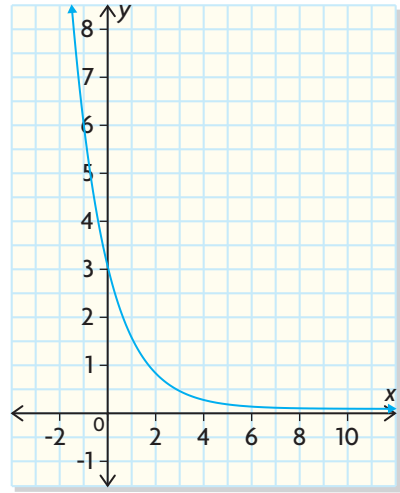
1. For each of the exponential functions shown state:

i) the number of x-intercepts
ii) the y-intercept
iii) the end behaviour
iv) the domain
v) the range
vi) whether the function is increasing or decreasing

a)



b)



2. Predict the characteristics of each exponential function. Record your predictions in a table like the one below. Verify your predictions using graphing technology.

a) $y = 5(3)^x$ **b)** $y = 30(0.1)^x$

Number of x-Intercepts	y-Intercept	End Behaviour	Domain	Range	Increasing or Decreasing

3. The population of the regional municipality of Wood Buffalo, Alberta, has grown rapidly since the discovery of one of the world's largest oil deposits. Its population in 1996 was 35 000 and has grown since at an annual rate of 8%.

a) Create a table of values that shows the population of Wood Buffalo from 1996 to 2006.

b) How do you know that an exponential function can be used to model this context?

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c) The exponential function that models the population growth of Wood Buffalo is $y = 35\,000(1.08)^x$, where x represents the time in years and y represents the population. Determine the y-intercept of this function. What does it represent in this context?

d) Graph the function. Use the graph to estimate

i) The population in 2011.

ii) When the population exceeded 100 000.

4. A 200 g sample of radioactive polonium-210 has a half-life of 138 days. The mass of polonium, $M(t)$, in grams, that remains after t days can be modelled by the exponential function

$$M(t) = 200\left(\frac{1}{2}\right)^{\frac{t}{138}}$$

a) Create a graph of the function.

b) Determine the mass that remains after 5 years.

c) Determine the length of time needed for the sample to decay to 50 g.

5. Solve each of the following.

a) $5^{2-x} = \frac{1}{125}$

b) $9^{x+5} = 27^x$

c) $4^{2x+1} = 8$

d) $6^{x+2} = 36^{x-2}$

6. Solve each of the following. Round your answer to two decimal places.

a) $8^{2x} = 100$

b) $3^{2x} = 7^{3x-1}$

7. A biologist tracks the population of an insect in a controlled environment over several years.

Year (t)	0	1	2	3	4	5
Population $P(t)$	160	192	275	315	405	479

a) Create a scatter plot for the data.

b) Determine the equation of the exponential regression function that models the insect's population growth.

c) Explain how the parameters in the equation are related to the context.

d) Determine when the population will exceed 1000.

8. If you invested \$1000 in an account that pays 5%/a compounded weekly, how many years would it take for your deposit to double?

9. Determine each value of y. Round to two decimal places where necessary.

a) $y = \ln 27$

b) $y = 2\log_3 27$

c) $y = \log_4 \frac{1}{64}$

d) $y = -\log 30$

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10. Predict the characteristics of each logarithmic function by stating

- the x-intercept
- the number of y-intercepts
- the end behaviour
- the domain
- the range
- whether the function is increasing or decreasing.

Verify your predictions by graphing each function.

a) $y = 30 \log x$ **b)** $y = -20 \ln x$

11. Evaluate each of the following.

a) $\log_5 625 + \log_2 32$

b) $\log_3 \frac{1}{27}$

c) $\log_2 4^5$

d) $\log_4 (64)^5$

e) $\log_4 8 + \log_4 \frac{1}{8}$

f) $\log_2 160 - \log_2 5$

12. Write each of the following as a single logarithm. Using the change of base formula, evaluate each logarithm to two decimal places.

a) $4 \log_4 2 + \log_4 6$ **b)** $\log_2 75 - 2 \log_2 5$

13. Explain how you can use the common logarithm to solve exponential equations. Use the equation $25 = 4^x$ in your explanation.

14. The population of a town is increasing at a rate of 5.2% per year. The city council believes they will have to add another elementary school when the population reaches 100 000. There are currently 65 200 people living in the town. How long do they have before the new school will be needed?

15. Martin is a fruit grower in the Okanagan Valley. He has planted and tracked the growth of a new variety of cherry tree he is considering planting on 10 acres of his farm.

Age of Tree (years)	Height (feet)	Age of Tree (years)	Height (feet)
1	5.0	7	18.8
2	9.2	8	19.0
3	13.1	9	19.3
4	15.0	10	19.7
5	16.8	11	20.0
6	17.1	12	20.8

a) Create a scatter plot for the data.

b) Determine the equation of the logarithmic regression function that models the tree's growth.

c) Determine the height of a tree of this variety when it is 15 years old.

d) Determine the age of a tree of this variety when it is 12 feet tall.

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16. **a)** Name these common characteristics of the graphs of

$$y = \sin x \text{ and } y = \cos x:$$

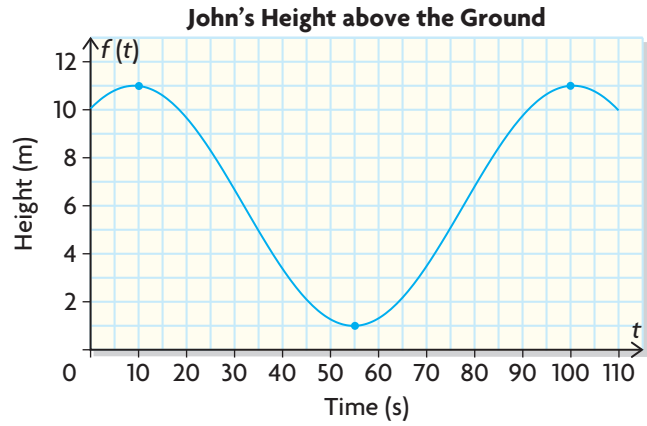
- the number of x-intercepts
- the y-intercept
- the domain
- the range
- the period
- the amplitude
- the equation of the midline

b) Sketch the graphs of both functions on the same axis where $\{x \mid 0 \leq x \leq 4\pi, x \in \mathrm{R}\}$.

17. A group of students is tracking a friend, John, as he rides a ferris wheel. They created a graph that shows his height above the ground at various times on his ride.

a) Describe this graph by determining its range, the equation of its midline, its amplitude, and its period.

b) Discuss how these characteristics of the graph are related to the ferris wheel.



18. For each of the following sinusoidal functions, state the amplitude, the period, the equation of the midline, the domain, the range, and the horizontal translation.

a) $y = 3 \sin 4(x - 30°) + 2$ **b)** $y = 5 \cos (x + 4) - 2$

19. The average high temperature for each month in Vancouver, British Columbia, is shown in the following table. Month 1 represents January and Month 12 represents December.

a) Create a scatter plot for the data. Why does assuming that this pattern repeats year after year make sense?

b) Determine the equation of the sinusoidal regression function that models the relationship between the month of the year and average high temperature.

c) Determine the average high temperature in Vancouver in the middle of July.

d) Estimate when the average high temperature in Vancouver will be greater than 60 °F.

Month Number	Average High Temperature (°F)
1	42
2	46
3	49
4	54
5	61
6	66
7	71
8	71
9	65
10	56
11	48
12	43

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